Symplectic groupoids in Poisson geometry

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Abstract

A symplectic groupoid is a Lie groupoid, endowed with a symplectic form that is compatible with the groupoid multiplication. The aim of present exposition is to show that symplectic groupoids generalize the usual symplectic structures, and that we can associate to any given symplectic groupoid, a canonical Poisson structure on the base manifold. Roughly speaking, we will see that we can interpret symplectic groupoids as integrated counterparts of Poisson structures, and the latter as infinitesimal counterparts of the former. To establish the fundamental results on symplectic groupoids, we first explore the notion of a symplectic realization of a Poisson structure, and provide a short introduction to Lie groupoids and algebroids. At last, we provide a recipe on how to construct the symplectic groupoid realization of a given Poisson structure, and demonstrate its use on a couple of examples.

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1 Poisson manifolds

In this first section, we recall the basic definitions and results on Poisson manifolds, and establish an equivalent formulation of a Poisson structure using bivector fields, as this is more useful for our purpose. Throughout this chapter, M will denote a smooth manifold.

Definition 1.1. A *Poisson structure* on M is a Lie bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$, which satisfies the *Leibniz rule*

$$\{f, gh\} = \{f, g\}h + g\{f, h\},$$

for all $f, g, h \in C^\infty(M)$. The pair $(M, \{\cdot,\cdot\})$ is then called a *Poisson manifold*. A *Poisson map* between Poisson manifolds $(M, \{\cdot,\cdot\}_M)$ and $(N, \{\cdot,\cdot\}_N)$ is a smooth map $\phi\colon M \to N$ that preserves the Poisson structures, that is:

$$\{f \circ \phi, g \circ \phi\}_M = \{f, g\}_N \circ \phi,$$

for all $f, g \in C^\infty(M)$.

Due to the Leibniz rule, we have that for any $f \in C^\infty(M)$, the map $\{f,\cdot\}\colon C^\infty(M) \to C^\infty(M)$ is a derivation on $C^\infty(M)$, so it corresponds to a unique vector field $X_f \in \mathfrak{X}(M)$, called the *Hamiltonian vector field* of f, defined implicitly as

$$\{f, g\} = \mathrm{d}g(X_f).$$

In a local chart $(U, (x^i)_i)$ on M, the Hamiltonian vector field X_f has the form

$$X_f = \sum_i X^i_f \partial_i$$

and now the Leibniz rule implies $X^i_{fg} = gX^i_f + fX^i_g$, hence the map $f|_U \mapsto X^i_f$ is also a derivation on $C^\infty(U)$, so $X^i_f = \sum_j \pi^{ij}\partial_j f$ for some functions $\pi^{ij} \in C^\infty(U)$. This implies that locally,

$$\{f, g\} = \sum_{i,j} \pi^{ij}(\partial_i f)(\partial_j g) \tag{1}$$

for some functions π^{ij}, and clearly we must have $\pi^{ij} = \{x^i, x^j\}$. These functions are called *structure functions* of the Poisson structure $\{\cdot,\cdot\}$ in the chart $(U, (x^i)_i)$. It is easy to see that the Jacobi identity on U is equivalent to the system of first-order partial differential equations:

$$\sum_l \left(\pi^{il}\partial_l\pi^{jk} + \pi^{jl}\partial_l\pi^{ki} + \pi^{kl}\partial_l\pi^{ij}\right) = 0, \quad i < j < k. \tag{2}$$

On the other hand, it is straightforward to see that the Jacobi identity for $\{\cdot,\cdot\}$ is equivalent to the map $f \mapsto X_f$ being Lie bracket preserving, that is:

$$X_{\{f,g\}} = [X_f, X_g]$$

for any two functions $f, g \in C^\infty(M)$.

Example 1.2.

(i) Any smooth manifold M admits a trivial Poisson structure $\{\cdot,\cdot\} = 0$.

(ii) Any symplectic structure ω on M gives rise to the Poisson structure on M, given by

$$\{f, g\} = -\omega(X_f, X_g),$$

where X_f is the unique vector field satisfying $\mathrm{d}f = \omega(X_f, \cdot)$. The minus in above definition makes sure that X_f is the Hamiltonian vector field of f with respect to the Poisson structure $\{\cdot,\cdot\}$ we have just defined, so that the two notions of a Hamiltonian vector field (with respect to a symplectic and a Poisson structure) agree. In Darboux coordinates $(q^i, p_i)_i$ on M, we have $\omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q^i$, so there holds $X_{q^i} = \partial_{p_i}, X_{p_i} = -\partial_{q^i}$. This means the structure functions are

$$\{p_i, q^j\} = \delta_i^j, \quad \{q^i, q^j\} = \{p_i, p_j\} = 0,$$

and the Poisson bracket reads

$$\{f, g\} = \sum_i \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}.$$

(iii) *Linear Poisson structures.* On $M = \mathbb{R}^n$ with the standard coordinates $(x^i)_i$, declaring the structure functions to be constant (that is, $\pi^{ij} = -\pi^{ji} \in \mathbb{R}$), always yields a Poisson structure trivially by equation (2).

(iv) *LV-type Poisson structures.* On $M = \mathbb{R}^n$ with the standard coordinates $(x^i)_i$, declaring the structure functions as $\pi^{ij} = a^{ij}x^i x^j$, where $a^{ij} = -a^{ji} \in \mathbb{R}$ are constant, yields a Poisson structure – we leave it to the reader to check that equation (2) holds.

1.1 Poisson bivector fields

The local expression (1) for the Poisson bracket shows that we may locally express it as a bivector field $\pi = \sum_{i,j} \pi^{ij}\partial_i \otimes \partial_j$ on the chart domain. To develop a global theory for Poisson brackets, it is crucial that we establish the bijective correspondence between Poisson brackets and Poisson bivector fields. To do so, we start with the following dual notion to differential forms.

Definition 1.3. A *multivector field* of degree k on M is a section of the vector bundle $\wedge^k TM$, i.e. it is an alternating $C^\infty(M)$-multilinear map

$$\vartheta\colon \prod_{i=1}^{k} \Omega^1(M) \to C^\infty(M).$$

We denote the $C^\infty(M)$-module of all k-vector fields by $\mathfrak{X}^k(M) = \Gamma^\infty\left(\wedge^k TM\right)$.

Remark 1.4. Defining $\mathfrak{X}^0(M) = C^\infty(M)$, it is a fact from multilinear algebra that the set

$$\mathfrak{X}^\bullet(M) = \bigoplus_{k=0}^{\dim M} \mathfrak{X}^k(M),$$

is an exterior algebra with respect to the the wedge product $\wedge\colon \mathfrak{X}^k(M) \times \mathfrak{X}^l(M) \to \mathfrak{X}^{k+l}(M)$,

$$(\vartheta \wedge \zeta)(\alpha_1, \ldots, \alpha_{k+l}) = \sum_{\sigma \in S_{k+l}} (-1)^{\operatorname{sgn}\sigma} \vartheta\left(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)}\right) \zeta\left(\alpha_{\sigma(k+1)}, \ldots, \alpha_{\sigma(k+l)}\right).$$

Moreover, any $\vartheta \in \mathfrak{X}^k(M)$ determines a map

$$\mathcal{L}_\vartheta \colon \prod_{i=1}^{k} C^\infty(M) \to C^\infty(M), \quad \mathcal{L}_\vartheta(f_1, \dots, f_k) = \vartheta(\mathrm{d}f_1, \dots, \mathrm{d}f_k),$$

which is clearly an alternating multiderivation (the latter means Leibniz rule holds in any of its arguments); that the map $\vartheta \mapsto \mathcal{L}_\vartheta$ is a bijection onto the set of all alternating multiderivations, is a well-known fact from the theory of smooth manifolds.

The last remark enables us to establish the wanted correspondence – in the particular case of degree 2, any Poisson structure $\{\cdot, \cdot\}$ on M is an alternating biderivation, hence there exists a bivector field $\pi \in \mathfrak{X}^2(M)$, defined by the equality $\pi(\mathrm{d}f, \mathrm{d}g) = \{f, g\}$, for any $f, g \in C^\infty(M)$. To rewrite the Jacobi identity as an identity for bivectors, note that for any $\vartheta \in \mathfrak{X}^2(M)$, the *Schouten bracket** $[\vartheta, \vartheta] \in \mathfrak{X}^3(M)$ is the unique 3-vector field, determined by the equation

$$\frac{1}{2}\mathcal{L}_{[\vartheta,\vartheta]}(f, g, h) = \mathcal{L}_\vartheta(f, \mathcal{L}_\vartheta(g, h)) + \mathcal{L}_\vartheta(g, \mathcal{L}_\vartheta(h, f)) + \mathcal{L}_\vartheta(h, \mathcal{L}_\vartheta(f, g)).$$

Hence, if π is a bivector field corresponding to an alternating biderivation $\{\cdot, \cdot\}$, then $[\pi, \pi] = 0$ holds if and only if the Jacobi identity for $\{\cdot, \cdot\}$ holds. We have just proven the following.

Proposition 1.5. Poisson structures on M are in a bijective correspondence with bivector fields $\pi \in \mathfrak{X}^2(M)$, satisfying $[\pi, \pi] = 0$. The bijection is induced by the map $\pi \mapsto \mathcal{L}_\pi$, so it is given by

$$\pi(\mathrm{d}f, \mathrm{d}g) = \{f, g\} \tag{3}$$

for any $f, g \in C^\infty(M)$.

From now on, a bivector field $\pi \in \mathfrak{X}^2(M)$ satisfying $[\pi, \pi] = 0$ will be called a *Poisson bivector field*, and sometimes also just a *Poisson structure*.

1.2 Vector bundle maps induced by Poisson structures

We now obtain for our purpose the most important way of seeing Poisson structures, by noting that any bivector field $\pi \in \mathfrak{X}^2(M)$ induces the following morphism of vector bundles:

$$\pi^\sharp \colon T^*M \to TM, \quad \alpha \mapsto \pi(\alpha, \cdot).$$

This morphism additionally satisfies the equality $\alpha(\pi^\sharp(\beta)) = -\beta(\pi^\sharp(\alpha))$, for any $\alpha, \beta \in \Omega^1(M)$; this just means that bivector fields are in a bijective correspondence with *skew-symmetric* vector bundle maps $\theta \colon T^*M \to TM$, i.e. the algebraic adjoint† θ^* satisfies

$$\theta^* = -\theta,$$

*The Schouten bracket is more generally defined as the map $\mathfrak{X}^k(M) \times \mathfrak{X}^l(M) \ni (\vartheta, \zeta) \mapsto [\vartheta, \zeta] \in \mathfrak{X}^{k+l-1}(M)$, defined as $\mathcal{L}_{[\vartheta,\zeta]} = \mathcal{L}_\vartheta \circ \mathcal{L}_\zeta + (-1)^{kl}\mathcal{L}_\zeta \circ \mathcal{L}_\vartheta$, where

$$(\mathcal{L}_\vartheta \circ \mathcal{L}_\zeta)(f_1, \dots, f_{k+l-1}) = \sum_{\sigma \in S_{k-1,l}} (-1)^{\deg \sigma} \mathcal{L}_\vartheta\big(f_{\sigma(1)}, \dots, f_{\sigma(k-1)}, \mathcal{L}_\zeta(f_{\sigma(k)}, \dots, f_{\sigma(k+l-1)})\big).$$

Here, $S_{k,l} = \{\sigma \in\subset S_{k+l} \mid \sigma(1) < \dots < \sigma(k) \text{ and } \sigma(k+1) < \dots < \sigma(k+l)\}$ denotes the set of (k,l)-shuffles.

†Recall that if $\theta \colon W \to V$ is a linear map, its algebraic adjoint is $\theta^* \colon V^* \to W^*$, $\alpha \mapsto \alpha \circ \theta$. In our case, $W = V^*$, and so $\theta^*(\alpha)(\beta) = \alpha(\theta(\beta))$

or equivalently $\alpha(\theta(\beta)) = -\beta(\theta(\alpha))$ for any two 1-forms $\alpha, \beta \in \Omega^1(M)$.

To express the condition $[\pi, \pi] = 0$ in terms of the bundle morphism $\pi^\sharp$, it turns out we have to consider the following bracket on the space of 1-forms:

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi^\sharp\alpha}\beta - \mathcal{L}_{\pi^\sharp\beta}\alpha - \mathrm{d}(\pi(\alpha, \beta)), \tag{4}$$

where $\mathcal{L}$ denotes the Lie derivative of a differential form along a vector field.

Proposition 1.6. Let $\pi \in \mathfrak{X}^2(M)$ be a bivector field, and let $\{f, g\} = \pi(\mathrm{d}f, \mathrm{d}g)$ be the associated alternating biderivation. The bracket $[\cdot, \cdot]_\pi$ is the unique $\mathbb{R}$-bilinear, alternating map $\Omega^1(M) \times \Omega^1(M) \to \Omega^1(M)$, such that the following holds:

(i) On exact forms, $[\mathrm{d}f, \mathrm{d}g]_\pi = \mathrm{d}\{f, g\}$ for all $f, g \in C^\infty(M)$.

(ii) *Leibniz rule:* $[\alpha, f\beta]_\pi = f[\alpha, \beta]_\pi + \mathrm{d}f(\pi^\sharp(\alpha))\beta$ for all $\alpha, \beta \in \Omega^1(M)$ and $f \in C^\infty(M)$.

Moreover, the following are equivalent:

(a) $[\pi, \pi] = 0$.

(b) The map $\pi^\sharp \colon (\Omega^1(M), [\cdot, \cdot]_\pi) \to (\mathfrak{X}^1(M), [\cdot, \cdot])$ preserves the brackets.

(c) $[\cdot, \cdot]_\pi$ satisfies the Jacobi identity.

Remark 1.7. We will later see that the properties (ii) and (b), and (c) are precisely those which make the triple $(T^*M, [\cdot, \cdot]_\pi, \pi^\sharp)$ into a Lie algebroid; this is the importance of $\pi^\sharp$ for our discussion.

Proof. Bilinearity and antisymmetry of $[\cdot, \cdot]_\pi$ is clear. For (i), note that for any $X \in \mathfrak{X}(M)$,

$$(\mathcal{L}_{\pi^\sharp \mathrm{d}f}\, \mathrm{d}g)(X) = (\pi^\sharp \mathrm{d}f)(Xg) - [\pi^\sharp \mathrm{d}f, X]g = X((\pi^\sharp \mathrm{d}f)g) = X\{f, g\},$$

and so $\mathcal{L}_{\pi^\sharp \mathrm{d}f}\, \mathrm{d}g = \mathrm{d}\{f, g\}$, and similarly $\mathcal{L}_{\pi^\sharp \mathrm{d}g}\, \mathrm{d}f = \mathrm{d}\{g, f\} = -\mathrm{d}\{f, g\}$. Hence (i) holds.

For (ii), note that

$$\mathcal{L}_{f\pi^\sharp\beta}\alpha = f\mathcal{L}_{f\pi^\sharp\beta}\alpha + \underbrace{\alpha(\pi^\sharp\beta)}_{\pi(\beta,\alpha)}\mathrm{d}f, \quad \mathrm{d}(\pi(\alpha, f\beta)) = f\,\mathrm{d}(\pi(\alpha, \beta)) + \pi(\alpha, \beta)\,\mathrm{d}f$$

and the rightmost terms of these expressions cancel out. The Leibniz rule for the first term of (4), together with the remaining above terms, now gives us the desired equality (ii).

For uniqueness, note that any $\mathbb{R}$-bilinear, alternating operation on $\Omega^1(M)$ that satisfies the Leibniz rule (ii), is determined by its values on exact forms, hence if it satisfies (i), it must equal $[\cdot, \cdot]_\pi$.

Let's prove the second part. To prove (a)$\leftrightarrow$(b), we consider the map $U_\pi \colon \Omega^1(M) \times \Omega^1(M) \to \mathfrak{X}^1(M)$,

$$U_\pi(\alpha, \beta) = [\pi^\sharp\alpha, \pi^\sharp\beta] - \pi^\sharp([\alpha, \beta]_\pi),$$

which is $C^\infty(M)$-bilinear by Leibniz rule (ii). We now evaluate

$$\begin{aligned}
\mathrm{d}h(U_\pi(\mathrm{d}f, \mathrm{d}g)) &= \mathrm{d}h[\pi^\sharp \mathrm{d}f, \pi^\sharp \mathrm{d}g] - \mathrm{d}h(\pi^\sharp([\mathrm{d}f, \mathrm{d}g]_\pi)) \\
&= \mathrm{d}h[X_f, X_g] - \{\{f, g\}, h\} \\
&= X_f(\mathrm{d}h(X_g)) - X_g(\mathrm{d}h(X_f)) + \{h, \{f, g\}\} \\
&= \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\},
\end{aligned}$$

for any $f, g, h \in C^\infty(M)$. By $C^\infty(M)$-linearity, this means that $\gamma(U_\pi(\alpha, \beta)) = \frac{1}{2}[\pi, \pi](\alpha, \beta, \gamma)$ for all $\alpha, \beta, \gamma \in \Omega^1(M)$, and this is just the Jacobiator $J_{\{\cdot,\cdot\}}(\alpha, \beta, \gamma)$, implying our equivalence.

To prove (b)$\leftrightarrow$(c), we consider the Jacobiator of $[\cdot, \cdot]_\pi$,

$$J_{[\cdot,\cdot]_\pi}(\alpha, \beta, \gamma) = [\alpha, [\beta, \gamma]_\pi]_\pi + [\beta, [\gamma, \alpha]_\pi]_\pi + [\gamma, [\alpha, \beta]_\pi]_\pi.$$

A straightforward computation shows that

$$J_{[\cdot,\cdot]_\pi}(\alpha, \beta, f\gamma) = f J_{[\cdot,\cdot]_\pi}(\alpha, \beta, \gamma) + \mathrm{d}f(U_\pi(\alpha, \beta))\gamma,$$

which shows that (c) implies $U_\pi = 0$, hence (b). Conversely, if $U_\pi = 0$, then $J_{[\cdot,\cdot]_\pi}$ is $C^\infty(M)$-multilinear, and we clearly have $J_{[\cdot,\cdot]_\pi}(\mathrm{d}f, \mathrm{d}g, \mathrm{d}h) = \mathrm{d}(J_{\{\cdot,\cdot\}}(\alpha, \beta, \gamma))$, which is 0 by (a), so we conclude $J_{[\cdot,\cdot]_\pi} = 0$. ■

Remark 1.8. Note that since $\pi^\sharp(\mathrm{d}f) = X_f$, we must have that at every point $x \in M$, the image

$$\mathrm{im}(\pi^\sharp_x) = \{X_f|_x \mid f \in C^\infty(M)\} \tag{5}$$

of $\pi^\sharp_x$ is a subspace of T_xM, consisting of so-called *Hamiltonian directions*. In general, this does not yield a (regular) distribution on M since the rank of $\pi^\sharp$ may not be constant, and we instead get a so-called *singular distribution*. The last proposition shows that the Poisson condition $[\pi, \pi] = 0$ nevertheless ensures the involutivity of this singular distribution. The study of the associated singular foliation is an interesting subject on its own; we direct the interested reader to [1, Chapter 4], where it is shown that the leaves of this singular foliation are in fact symplectic manifolds.

Example 1.9. Any 2-form $\omega \in \Omega^2(M)$ determines the vector bundle map

$$\omega^\flat \colon TM \to T^*M, \quad v \mapsto \omega(v, \cdot).$$

In the case when ω is nondegenerate, the inverse $(\omega^\flat)^{-1}$ of $\omega^\flat$ is clearly a skew-symmetric map. Hence we must have $(\omega^\flat)^{-1} = \pi^\sharp$ for some unique $\pi \in \mathfrak{X}^2(M)$. It is easy to see that this can be rewritten as

$$\omega(\pi^\sharp\alpha, \pi^\sharp\beta) = -\pi(\alpha, \beta),$$

Using the formula for exterior derivative of 2-forms on the expression $\mathrm{d}\omega(\pi^\sharp\,\mathrm{d}f, \pi^\sharp\,\mathrm{d}g, \pi^\sharp\,\mathrm{d}h)$, we get

$$\begin{aligned}\mathrm{d}\omega(\pi^\sharp\,\mathrm{d}f, \pi^\sharp\,\mathrm{d}g, \pi^\sharp\,\mathrm{d}h) = {}& X_f(\omega(X_g, X_h)) + X_g(\omega(X_h, X_f)) + X_h(\omega(X_f, X_g)) \\ & - \omega([X_f, X_g], X_h) - \omega([X_g, X_h], X_f) - \omega([X_h, X_f], X_g)\end{aligned}$$

from which it is straightforward to see

$$\mathrm{d}\omega(\pi^\sharp\,\mathrm{d}f, \pi^\sharp\,\mathrm{d}g, \pi^\sharp\,\mathrm{d}h) = -2(\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}),$$

and hence by $C^\infty(M)$-linearity,

$$[\pi, \pi](\alpha, \beta, \gamma) = -\,\mathrm{d}\omega(\pi^\sharp\alpha, \pi^\sharp\beta, \pi^\sharp\gamma),$$

for all $\alpha, \beta, \gamma \in \Omega^1(M)$. This means that closedness of ω is equivalent to the Jacobi identity for π. Hence, there is a bijective correspondence between symplectic forms and nondegenerate Poisson bivector fields on M. ♦

1.3 Bivector fields and Poisson maps

Recall from the theory of smooth manifolds that, given a smooth map $\phi\colon M \to N$, a vector field $X \in \mathfrak{X}(M)$ is said to be *ϕ-related* to $Y \in \mathfrak{X}(N)$, if for any $x \in X$, there holds

$$Y_{\phi(x)} = \mathrm{d}\phi_x(X_x).$$

Definition 1.10. Let $\phi\colon M \to N$ be a smooth map. A k-vector field $\vartheta \in \mathfrak{X}^k(M)$ is *ϕ-related* to a k-vector field $\zeta \in \mathfrak{X}^k(N)$, if for any $x \in X$ there holds

$$\zeta_{\phi(x)} = (\mathrm{d}\phi_x)_*(\vartheta_x),$$

where the linear map $(\mathrm{d}\phi_x)_*\colon \bigwedge^k T_xM \to \bigwedge^k T_{\phi(x)}N$ is induced in the obvious way by $\mathrm{d}\phi_x$.

Remark 1.11. If ζ is ϕ-related to ϑ, we must have $\zeta_y = (\mathrm{d}\phi_x)_*(\vartheta_x)$ for every $x \in \phi^{-1}(y)$ and every $y \in \operatorname{im}\phi$. If the map ϕ is also surjective, then ζ is completely determined by ϑ – in this case, we say that ζ is the *pushforward* of ϑ along ϕ, and write $\zeta = \phi_*\vartheta$.

A similar proof to that for ordinary vector fields shows that ϑ is ϕ-related to ζ if and only if there holds

$$\mathcal{L}_\vartheta(f_1 \circ \phi, \ldots, f_k \circ \phi) = \mathcal{L}_\zeta(f_1, \ldots, f_k) \circ \phi \tag{6}$$

for any $f_1, \ldots, f_k \in \mathbb{C}^\infty(N)$.

The notion of relatedness of k-vector fields enables us to establish the following characterization of Poisson maps, which will be used in our discussion.

Proposition 1.12. Let $\phi\colon M \to N$ be a smooth map between Poisson manifolds (M, π_M) and (N, π_N). The following are equivalent:

(i) ϕ is a Poisson map.

(ii) π_M is ϕ-related to π_N.

(iii) For any $f \in C^\infty(N)$ the Hamiltonian vector fields $X_{f\circ\phi} \in \mathfrak{X}(M)$ and $X_f \in \mathfrak{X}(N)$ are ϕ-related.

(iv) $\pi_N^\sharp = \mathrm{d}\phi_x \circ \pi_M^\sharp \circ (\mathrm{d}\phi_x)^*$, i.e. the following diagram commutes:

$$\begin{tikzcd}
T_xM \arrow[r, "\mathrm{d}\phi_x"] & T_{\phi(x)}N \\
T_x^*M \arrow[u, "\pi_M^\sharp"] & T_{\phi(x)}^*N \arrow[l, "(\mathrm{d}\phi_x)^*"] \arrow[u, "\pi_N^\sharp"']
\end{tikzcd}$$

Proof. The characterizing equation (6) of ϕ-relatedness trivially gives equivalences (i)↔(ii) and (i)↔(iii). To show the equivalence (ii)↔(iv), note that for any $\alpha, \beta \in T^*_{\phi(x)}N$, (ii) can be rewritten as

$$\begin{aligned}
\big((\mathrm{d}\phi_x)_*\pi_M\big)(\alpha, \beta) &= \pi_N(\alpha, \beta) \\
\big(\,\mathrm{d}\phi_x{}^*\beta\big)(\pi_M^\sharp(\mathrm{d}\phi_x{}^*\alpha)) &= \beta(\pi_N^\sharp(\alpha)) \\
\beta(\mathrm{d}\phi_x(\pi_M^\sharp(\mathrm{d}\phi_x{}^*\alpha))) &= \beta(\pi_N^\sharp(\alpha)).
\end{aligned}$$

Since this holds for all $\alpha, \beta \in T^*_{\phi(x)}N$, we are done. ∎

2 Symplectic realizations

As observed in Remark 1.8, a Poisson structure can have non-constant rank, and will not be well-behaved globally, in general. An important way to study such behaviour is by considering nondegenerate Poisson structures, which are submersed onto our Poisson manifold of interest, via a Poisson map.

Definition 2.1. A *symplectic realization* of a Poisson manifold (M, π) is a symplectic manifold (S, ω), together with a surjective submersion $\mu \colon S \to M$ that is a Poisson map.

In the last definition, we are endowing S with the nondegenerate Poisson structure π_ω from Example 1.9, i.e. $\pi_\omega^\sharp = (\omega^\flat)^{-1}$. Proposition 1.12 shows that the requirement on μ of being a Poisson map can be expressed as

$$\pi^\sharp|_{\mu(p)} = \mathrm{d}\mu_p \circ \pi_\omega^\sharp|_p \circ (\mathrm{d}\mu_p)^* \tag{7}$$

for any $p \in S$. Furthermore, given any $p \in S$, notice that since $\mathrm{d}\mu_p$ is surjective, $(\mathrm{d}\mu_p)^*$ is injective, hence $\pi_\omega^\sharp \circ (\mathrm{d}\mu_p)^*$ is also injective, and it maps $\ker(\pi^\sharp|_{\mu(p)})$ into $\ker(\mathrm{d}\mu_p)$. Hence by rank-nullity,

$$\dim M - \operatorname{rank} \pi_{\mu(p)} = \dim(\ker(\pi^\sharp|_{\mu(p)})) \leq \dim(\ker(\mathrm{d}\mu_p)) = \dim S - \dim M,$$

and since μ is surjective, the following proposition is proven.

Proposition 2.2. If (S, ω) is a symplectic realization of (M, π), then for any $x \in M$, we have

$$\dim S \geq 2 \dim M - \operatorname{rank} \pi_x.$$

Example 2.3. Consider the zero Poisson structure $\{x, y\} = 0$ on $\mathbb{R}^2$. By last proposition, we must have $\dim S \geq 4$, so we try with $S = \mathbb{R}^4$, and denote the variables on $\mathbb{R}^4$ as (x, y, u, v). A simple way of obtaining a nondegenerate Poisson structure on $\mathbb{R}^4$ is by adding to $\{x, y\} = 0$ the structure functions

$$\{x, u\} = 1, \quad \{y, v\} = 1, \quad \{u, v\} = 0.$$

In this way, we obtain the canonical symplectic structure on $\mathbb{R}^4$, together with the map

$$\mu(x, y, u, v) = (x, y).$$

In the last section, this example will be generalized to $\mathbb{R}^{2n}$ in a more insightful way. ♦

Remark 2.4. Given a Poisson manifold (M, π), the recipe for obtaining a symplectic realization locally has the following form. If $\mu \colon (S, \omega) \to (M, \pi)$ is a symplectic realization, and $x = (x^1, \ldots, x^m)$ are local coordinates on M with structure functions $\{x^i, x^j\} = \pi^{ij}(x)$, then there are (by rank theorem) local coordinates $(x, u) = (x^1, \ldots, x^m, u_1, \ldots, u^n)$ on S, such that the map μ is given by the projection. Since μ is a Poisson map, the structure functions of the nondegenerate Poisson structure on S read

$$\{x^i, x^j\} = \pi^{ij}(x), \quad \{x^i, u^a\} = \vartheta^{ia}(x, u), \quad \{u^a, u^b\} = \varphi^{ab}(x, u),$$

or more explicitly,

$$\pi_\omega = \sum_{i<j} \pi^{ij}(x) \partial_{x^i} \wedge \partial_{x^j} + \sum_{i<a} \vartheta^{ij}(x, u) \partial_{x^i} \wedge \partial_{u^a} + \sum_{a<b} \varphi^{ab}(x, u) \partial_{u^a} \wedge \partial_{u^b}.$$

Thus, when searching for a symplectic realization of a Poisson manifold which is covered by one chart, we should adjoin to coordinates $(x^i)_i$ the coordinates $(u^a)_a$ and find new functions $\vartheta^{ia}, \varphi^{ab}$ such that π_ω is a nondegenerate Poisson structure.

2.1 Libermann's theorem

As Remark 1.7 suggests and as we will later see, the questions of existence and uniqueness of a symplectic realization boil down to problems in Lie theory. The following theorem is an important step towards seeing this.

Theorem 2.5 (Libermann)**.** Let (S, ω) be a symplectic manifold and let $\mu \colon S \to M$ be a surjective submersion with connected fibres. Then M admits a Poisson structure π such that $\mu \colon (S, \omega) \to (M, \pi)$ is a symplectic realization, if and only if the ω-orthogonal distribution to the fibres of μ,

$$(\ker \mathrm{d}\mu)^\omega \subset TS,$$

is an involutive distribution. In this case, π is unique.

Proof. Equation (7) shows that π is unique, if it exists. To show existence, note that since $\mu \colon S \to M$ is a surjective submersion, any function $f \colon M \to \mathbb{R}$ is smooth if and only if $f \circ \mu \colon S \to \mathbb{R}$ is, hence we have an injective map

$$\mu^* \colon C^\infty(M) \hookrightarrow C^\infty(S), \quad f \mapsto f \circ \mu.$$

Denote by $C^\infty_M(S) = \mu^*(C^\infty(M))$ the image of μ^*. Since we require our map μ to be Poisson, injectivity of μ^* implies that existence of π is equivalent to:

$$f, g \in C^\infty_M(S) \text{ implies } \{f, g\} \in C^\infty_M(S).$$

Furthermore, since μ has connected fibres, we can write

$$C^\infty_M(S) = \{f \in C^\infty(S) \mid \mathrm{d}f(V) = 0 \text{ for all } V \in \Gamma^\infty(\ker \mathrm{d}\mu)\},$$

i.e. any function $f \in C^\infty_M(S)$ must be constant on the fibres of μ. For any $f \in C^\infty_M(S)$,

$$0 = \mathrm{d}f(V) = \omega(X_f, V) \text{ for all } V \in \Gamma^\infty(\ker \mathrm{d}\mu),$$

hence $f \in C^\infty_M(S)$ if and only if $X_f \in \Gamma^\infty((\ker \mathrm{d}\mu)^\omega)$. Since $X_{\{f,g\}} = [X_f, X_g]$ for any $f, g \in C^\infty(S)$, we see that the existence of π is equivalent to:

$$X_f, X_g \in \Gamma^\infty((\ker \mathrm{d}\mu)^\omega) \text{ implies } [X_f, X_g] \in \Gamma^\infty((\ker \mathrm{d}\mu)^\omega). \tag{8}$$

Now for this to hold, it is sufficient that $(\ker \mathrm{d}\mu)^\omega$ is an involutive distribution. To show that this is also necessary, suppose (8) holds, so now we must check that

$$X, Y \in \Gamma^\infty((\ker \mathrm{d}\mu)^\omega) \text{ implies } [X, Y] \in \Gamma^\infty((\ker \mathrm{d}\mu)^\omega).$$

To see this, we first note that the map

$$\begin{aligned} &\Gamma^\infty((\ker \mathrm{d}\mu)^\omega) \times \Gamma^\infty((\ker \mathrm{d}\mu)^\omega) \to \Gamma^\infty\left(\frac{TS}{(\ker \mathrm{d}\mu)^\omega}\right) \\ &(X, Y) \mapsto [X, Y] \text{ mod } (\ker \mathrm{d}\mu)^\omega \end{aligned}$$

is $C^\infty(S)$-bilinear (this holds for any subbundle of TS), hence corresponds to a unique bundle morphism

$$\xi \colon (\ker \mathrm{d}\mu)^\omega \oplus (\ker \mathrm{d}\mu)^\omega \to \frac{TS}{(\ker \mathrm{d}\mu)^\omega},$$

which clearly vanishes if and only if $(\ker \mathrm{d}\mu)^\omega$ is involutive. So it suffices to show our claim pointwise, i.e. that for any $p \in S$ and $v \in (\ker \mathrm{d}\mu_p)^\omega$ there is a function $f \in C^\infty_M(S)$ with $v = X_f|_p$ (indeed, condition (8) then implies that the bundle morphism ξ vanishes at p). To see this, note that we have a composition of isomorphisms

$$(\ker \mathrm{d}\mu_p)^\omega \xrightarrow{\omega^\flat} (\ker \mathrm{d}\mu_p)^\circ \xleftarrow{(\mathrm{d}\mu_p)^*} T^*_{\mu(p)}M \tag{9}$$

where $(\ker \mathrm{d}\mu_p)^\circ = \{\alpha \in T^*_pS \mid \alpha(v) = 0 \text{ for all } v \in \ker \mathrm{d}\mu_p\}$. Under this isomorphism, $v \in (\ker \mathrm{d}\mu_p)^\omega$ corresponds to $\mathrm{d}g_{\mu(p)}$ for some $g \in C^\infty(M)$, and now $f = \mu^*g = g \circ \mu$ has the desired property $v = X_f|_p$. ■

Remark 2.6. If $\mu\colon (S,\omega) \to (M,\pi)$ is a symplectic realization, then together with equation (5), the last step in the proof shows

$$\mathrm{d}\mu_p((\ker \mathrm{d}\mu_p)^\omega) = \mathrm{im}\left(\pi^\sharp|_{\mu(p)}\right),$$

so that the (regular) symplectic orthogonal distribution to $\ker \mathrm{d}\mu$ is pushed forward along μ to the possibly singular distribution, consisting of Hamiltonian directions.

2.2 Infinitesimal actions of symplectic realizations

By Libermann's theorem, a symplectic realization $\mu\colon (S,\omega) \to (M,\pi)$ comes with two distinguished involutive distributions on TS, the *vertical distribution* $\ker \mathrm{d}\mu$ and the *orbit distribution* $(\ker \mathrm{d}\mu)^\omega$. By Frobenius' theorem, each of them is integrable, so they respectively give rise to a *vertical foliation* and a *orbit foliation* of S.

The name "orbit" comes from the fact that it arises from the following action.

Definition 2.7. The *infinitesimal action* of a symplectic realization $\mu\colon (S,\omega) \to (M,\pi)$ is the map $\mathfrak{a}\colon \Omega^1(M) \to \mathfrak{X}^1(S)$, defined by

$$\omega(\mathfrak{a}(\alpha), \cdot) = \mu^*\alpha.$$

Remark 2.8. In the case $\alpha = \mathrm{d}f$ for some $f \in C^\infty(M)$, $\mathfrak{a}(\alpha)$ is just the Hamiltonian vector field $X_{f\circ\mu}$ on S.

Since this is a C^∞-linear map in the sense that $\mathfrak{a}(f\alpha) = (f \circ \mu)\mathfrak{a}(\alpha)$ for any $f \in C^\infty(M)$, this map defines a vector bundle morphism $\mathfrak{a}\colon \mu^*(T^*M) \to TS$.

To justify the nomenclature, we now show that on the level of sections, $\mathfrak{a}$ is a Lie algebra morphism, and that its image is the orbit distribution.

Proposition 2.9. The infinitesimal action $\mathfrak{a}\colon \Omega^1(M) \to \mathfrak{X}^1(S)$ of a symplectic realization $\mu\colon (S,\omega) \to (M,\pi)$ is a morphism of Lie algebras, that is,

$$\mathfrak{a}([\alpha,\beta]_\pi) = [\mathfrak{a}(\alpha), \mathfrak{a}(\beta)]$$

for any $\alpha, \beta \in \Omega^1(M)$. It satisfies the following properties at any $p \in S$:

(i) The map $\mathfrak{a}$ lifts the map $\pi^\sharp$, i.e. we have the following diagram:

$$\begin{tikzcd} & T_pS \arrow[d, "\mathrm{d}\mu_p"] \\ T^*_{\mu(p)}M \arrow[ur, "\mathfrak{a}_p"] \arrow[r, "\pi^\sharp_{\mu(p)}"'] & T_{\mu(p)}M \end{tikzcd}$$

(ii) The map $\mathfrak{a}_p$ is injective.

(iii) The image of $\mathfrak{a}_p$ is the orbit distribution: $\mathrm{im}(\mathfrak{a}_p) = (\ker \mathrm{d}\mu_p)^\omega$.

(iv) The restriction of $\mathfrak{a}_p$ to $\ker \pi^\sharp_{\mu(p)}$ is an isomorphism onto $\ker \mathrm{d}\mu_p \cap (\ker \mathrm{d}\mu_p)^\omega$.

Proof. We first prove the properties (i)-(iv). Property (i) holds since μ is a Poisson map, so we may augment the above diagram to:

$$\begin{tikzcd} T_p^*S \arrow[r, "(\omega_p^\flat)^{-1}"] & T_pS \arrow[d, "\mathrm{d}\mu_p"] \\ T^*_{\mu(p)}M \arrow[u, "(\mathrm{d}\mu_p)^*"] \arrow[ur, dashed, "\mathfrak{a}_p"] \arrow[r, "\pi^\sharp_{\mu(p)}"'] & T_{\mu(p)}M \end{tikzcd}$$

This also shows that $\mathfrak{a}_p = (\omega_p^\flat)^{-1} \circ (\mathrm{d}\mu_p)^*$, which is a composition of an isomorphism with an injective map $(\mathrm{d}\mu_p)^*$ (since μ is a submersion), thus proving (ii). For (iii), observe that we have

$$\mathrm{im}(\mathfrak{a}_p) = (\omega_p^\flat)^{-1}(\mathrm{im}(\mathrm{d}\mu_p{}^*)) = (\omega_p^\flat)^{-1}((\ker \mathrm{d}\mu)^\circ) = (\ker \mathrm{d}\mu)^\omega,$$

as in the diagram (9). For (iv), note that since $\mathfrak{a}_p$ is injective by (ii), it maps $\ker \pi^\sharp_{\mu(p)}$ bijectively onto $\ker \mathrm{d}\mu_p \cap \mathrm{im}\, \mathfrak{a}_p$, so our wanted conclusion now follows from (iii).

To show $\mathfrak{a}$ is a Lie algebra morphism, we consider the map $A\colon \Omega^1(M) \times \Omega^1(M) \to \mathfrak{X}^1(S)$,

$$A(\alpha, \beta) = \mathfrak{a}([\alpha, \beta]_\pi) - [\mathfrak{a}\alpha, \mathfrak{a}\beta].$$

We have to check that $A = 0$. To do so, first compute

$$A(\alpha, f\beta) = (f \circ \mu)\mathfrak{a}([\alpha, \beta]_\pi) + \mathrm{d}f(\pi^\sharp\alpha)\mathfrak{a}\beta - (f \circ \mu)[\mathfrak{a}\alpha, \mathfrak{a}\beta] - \mathrm{d}(f \circ \mu)(\mathfrak{a}\alpha)\mathfrak{a}\beta\big)$$

where we have used $\mathfrak{a}(f\beta) = (f \circ \mu)\mathfrak{a}\beta$ and Leibniz rule for both brackets. The second and fourth term cancel out by (i), which can be written for 1-forms as $\mu_* \circ \mathfrak{a} = \pi^\sharp$. This implies that A is C^∞-linear in the second argument, but since A is alternating, this also holds for the first argument. Hence it is enough to check that $A(\mathrm{d}f, \mathrm{d}g) = 0$ for any $f, g \in C^\infty(M)$, but this is equivalent by Proposition 1.6 to

$$X_{\{f,g\}\circ\mu} = [X_{f\circ\mu}, X_{g\circ\mu}],$$

which holds by Proposition 1.12 since μ is a Poisson map. ■

3 Lie groupoids and algebroids

This section serves as an overview of the basics on Lie groupoids and algebroids.

3.1 Lie groupoids

Definition 3.1. A *groupoid* is a small category (i.e. the classes of its objects and morphisms are sets) where every morphism is invertible. More precisely, a groupoid consists of:

(i) A set G of morphisms and a set M of objects.

(ii) Two maps $s, t\colon G \to M$, called the *source* and *target* maps, which prescribe to any morphism its domain and codomain (respectively),

(iii) A *unit* map $u\colon M \to G$, which assigns to every object $x \in M$ the identity morphism $u(x) = 1_x = \mathrm{id}_x$, and an *inversion* map $\mathrm{inv}\colon G \to G$, which assigns to any morphism $g \in G$ its inverse g^{-1}.

(iv) A *partial multiplication* map $m\colon G * G \to G$, where $G * G = \{(g,h) \in G \times G \mid s(g) = t(h)\}$ is the set of pairs of *composable* morphisms, which sends $(g,h) \in G*G$ to their composition gh.

These maps must be such that, for any $g, h, k \in G$ and $x \in M$, there holds:

(i) $s(gh) = s(h)$ and $t(gh) = t(g)$, whenever $s(h) = t(g)$

(ii) $s(1_x) = t(1_x) = x$ and $g1_{s(g)} = 1_{t(g)}g = g$,

(iii) for any $g \in G$ there exists a unique $g^{-1} \in G$ such that $s(g^{-1}) = t(g)$, $t(g^{-1}) = s(g)$, $g^{-1}g = 1_{s(g)}$ and $gg^{-1} = 1_{t(g)}$,

(iv) $(gh)k = g(hk)$ whenever $s(g) = t(h)$ and $s(h) = t(k)$.

Given $x \in M$, we also define *s-fibre* over x as $G_x := s^{-1}(x)$, *t-fibre* over x as $G^x := t^{-1}(x)$, and the *vertex group* at x as $G_x^x := G_x \cap G^x$. Notice that the set of all morphisms from x to y is just $G_x^y := G_x \cap G^y$. We will sometimes denote a morphism $g \in G_x^y$ by $g\colon x \to y$.

A *Lie groupoid* is a groupoid with G and M smooth manifolds (G not necessarily Hausdorff), such that the maps s, t are smooth submersions, and u, inv and m are smooth – we say that the smooth structures on the groupoid are *compatible* with the groupoid structure.

Remark 3.2. We write $G \rightrightarrows M$ for a groupoid, to mean the whole structure:

$$\begin{tikzcd} G * G \arrow[r, "m"] & G \arrow[loop above, "\mathrm{inv}"] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & M \arrow[r, "u"] & G \end{tikzcd}$$

It is necessary to note that in order for $m\colon G * G \to G$ to be a smooth map, we must make sense of the smooth structure on $G * G$. To that end, we notice that $G * G = (s \times t)^{-1}(\Delta_M)$, where $\Delta_M = \{(x,x) \mid x \in M\} \subset M \times M$ is the diagonal, which is an embedded submanifold of $M \times M$. To show $G * G$ is a smooth submanifold of $G \times G$, we invoke transversality theorem; we must show

$$\mathrm{d}(s \times t)_{(g,h)}(T_gG \oplus T_hG) + T_{(x,x)}\Delta_M = T_{(x,x)}(M \times M),$$

for any $(g,h) \in G * G$, or equivalently

$$\mathrm{d}s_g(T_gG) \oplus \mathrm{d}t_h(T_hG) + \{(v,v) \mid v \in T_xM\} = T_xM \oplus T_xM,$$

but this clearly holds since s and t are submersions. Transversality theorem thus ensures that $G * G \subset G \times G$ is an embedded submanifold of dimension $\dim(G * G) = 2\dim G - \dim M$, and furthermore, that its tangent space at $(g, h) \in G * G$ is given as

$$T_{(g,h)}(G * G) = \mathrm{d}(s \times t)^{-1}_{(g,h)}\left(T_{(x,x)}\right) = \{(v, w) \in T_g G \oplus T_h G \mid \mathrm{d}s_g(v) = \mathrm{d}t_h(w)\}. \tag{10}$$

Example 3.3.

(i) *Trivial groupoid.* Let M be a smooth manifold and G a Lie group (G may not be acting on M). We define the trivial groupoid $M \times G \times M \rightrightarrows M$ with the product smooth structure, and the groupoid structure given by:

- s and t are projections to the third and first factor (resp.),
- the unit map is given by $1_x = (x, e, x)$ and inverse map by $(y, g, x)^{-1} = (x, g^{-1}, y)$,
- the partial multiplication is given by $(z, h, y)(y, g, x) = (z, hg, x)$.

It is straightforward to check that the groupoid axioms are satisfied, and that the smooth structure is compatible with the groupoid structure (it follows from that on G). Note that in the case when $M = \{*\}$ is a singleton, we can identify $\{*\} \times G \times \{*\} \rightrightarrows \{*\}$ with the Lie group G, so that groupoids are a generalization of groups. In case $G = \{e\}$ is a trivial group, we call the obtained Lie groupoid a *pair groupoid* and just write $M \times M \rightrightarrows M$.

(ii) *Action groupoid.* Let M be a smooth manifold and G a Lie group acting on it (from the right). We define a Lie groupoid $M \times G \rightrightarrows M$ by endowing it with the product smooth structure, and the compatible groupoid structure given by:

- $t(x, g) = x$, $s(x, g) = xg$,
- the unit map is given by $1_x = (x, e)$ and inverse map by $(x, g)^{-1} = (xg, g^{-1})$,
- the partial multiplication is given by $(x, g)(xg, h) = (x, gh)$.

We will write $M \rtimes G$ to mean the obtained Lie groupoid. It is easy to see that the t-fibres are $(M \rtimes G)^x = \{x\} \times G$, that the s-fibre of $M \rtimes G$ at x may be identified with the orbit $\mathrm{Orb}_G(x)$, and that the vertex group $(M \rtimes G)^x_x = \{(x, g) \mid xg = x\}$ at x may be identified with the stabilizer subgroup $\mathrm{Stab}_G(x)$ at x.

As a concrete example, the action of $\mathbb{R}$ on S^1 given by $(z, t) \mapsto \mathrm{e}^{2\pi i t} z$, provides us with a groupoid structure on the cylinder, with the t-fibre at z given by a vertical line $(S^1 \rtimes \mathbb{R})^z = \{z\} \times \mathbb{R}$, the s-fibre at z given by $(S^1 \rtimes \mathbb{R})_z = \{(w, t) \mid \mathrm{e}^{2\pi i t} w = z\}$ (visually depicted as a "spiral" on the cylinder), and $(S^1 \rtimes \mathbb{R})^z_z = \{z\} \times \mathbb{Z}$.

Note that in the case of a left action of G on M, the appropriate Lie groupoid is $G \times M \rightrightarrows M$, with the structure maps given by $s(g, x) = x$, $t(g, x) = gx$, partial multiplication given by $(h, gx)(g, x) = (hg, x)$ with the units $1_x = (e, x)$ and the inverse $(g, x)^{-1} = (g^{-1}, gx)$.

There are many more natural examples of Lie groupoids; among most important ones are the so-called gauge groupoids, which capture the structure of principal bundles (in particular, a fundamental groupoid is also a gauge groupoid).

3.2 Lie algebroids

Lie algebroids are vector bundles that possess a certain analogue of the usual Lie bracket. It is customary to see them as "oidified" versions of Lie algebras. As we will see, such structures naturally arise as infinitesimal counterparts of Lie groupoids.

Definition 3.4. A *Lie algebroid* on a smooth manifold M is a vector bundle $A \to M$, equipped with a Lie bracket $[\cdot,\cdot]\colon \Gamma^\infty(A) \times \Gamma^\infty(A) \to \Gamma^\infty(A)$ and a morphism of vector bundles $\rho\colon A \to TM$, such that the following holds:[‡]

(i) $\rho[\alpha,\beta] = [\rho\alpha, \rho\beta]$, for all $\alpha, \beta \in \Gamma^\infty(A)$.

(ii) *Leibniz rule.* $[\alpha, f\beta] = f[\alpha,\beta] + \rho(\alpha)(f)\beta$, for all $\alpha,\beta \in \Gamma^\infty(A)$ and $f \in C^\infty(M)$.

A *morphism* of Lie algebroids $A \to M$ and $B \to M$ is a vector bundle morphism $\Phi\colon A \to B$, such that:

(i) Φ preserves Lie brackets, i.e. $\Phi([\alpha,\beta]_A) = [\Phi\alpha, \Phi\beta]_B$ for all $\alpha,\beta \in \Gamma^\infty(A)$.

(ii) Φ preserves anchor maps, i.e. $\rho_B \circ \Phi = \rho_A$.

Example 3.5.

(i) Two trivial examples of Lie algebroids are the tangent bundle $(TM, [\cdot,\cdot], \mathrm{id}_{TM})$ of a smooth manifold M, and a Lie algebra, which is just the Lie algebroid over a singleton, with anchor equal to zero.

(ii) Proposition 1.6 shows that any Poisson manifold (M,π) gives rise to a Lie algebroid $(T^*M, [\cdot,\cdot]_\pi, \pi^\sharp)$, called the *Poisson algebroid* of (M,π).

Furthermore, Proposition 2.9 suggests the following definition, making the infinitesimal action of any symplectic realization an example thereof.

Definition 3.6. An *action* of a Lie algebroid $A \to M$ on a smooth map $\mu\colon S \to M$ is a Lie algebra morphism $\mathfrak{a}\colon \Gamma^\infty(A) \to \mathfrak{X}(S)$, such that:

(i) $\mathfrak{a}$ is C^∞-linear, i.e. $\mathfrak{a}(f\alpha) = (f \circ \mu)\mathfrak{a}(\alpha)$ for all $\alpha \in \Gamma^\infty(A)$ and $f \in C^\infty(M)$.

(ii) $\mathfrak{a}$ lifts the anchor, i.e. $\mathrm{d}\mu_p(\mathfrak{a}(\alpha)) = \rho(\alpha)_{\mu(p)}$ for any $\alpha \in \Gamma^\infty(A)$ and $p \in S$.

3.3 The Lie algebroid of a Lie groupoid

To "oidify" the construction of a Lie algebra on a Lie group, we recall that the latter is defined as the set

$$\mathfrak{X}_L(G) = \{X \in \mathfrak{X}(G) \mid (L_g)_* X = X \text{ for all } X \in G\},$$

of all left-invariant vector fields on G and accompanied with the Lie bracket. We also have a canonical isomorphism $\mathfrak{X}_L(G) \to T_eG = \mathfrak{g}$, $X \mapsto X_e$, with inverse $X_e \mapsto (g \overset{X}{\mapsto} \mathrm{d}(\mathcal{L}_g)_e(X_e))$.

Definition 3.7. A *left-invariant vector field* on a Lie groupoid $G \rightrightarrows M$ is a vector field $X \in \mathfrak{X}(G)$ that satisfies the following:

(i) X is tangent to t-fibres, i.e. $X \in \Gamma^\infty(\ker \mathrm{d}t)$.

(ii) $\mathrm{d}(L_g)_h(X_h) = X_{gh}$ for all $(g,h) \in G * G$.

The set of all left-invariant vector fields on $G \rightrightarrows M$ is denoted by $\mathfrak{X}_L(G)$.

The following is an "oidified" version of the result for Lie algebras of Lie groups.

[‡]It is left to the reader as an exercise to show that the property (i) in the definition of a Lie algebroid is actually redundant, as it is a consequence of Jacobi identity and Leibniz rule.

Lemma 3.8. On any Lie groupoid $G \rightrightarrows M$, the set $\mathfrak{X}_L(G)$ is closed under the Lie bracket on $\mathfrak{X}(G)$. As a vector space, it is isomorphic to the space of sections $\Gamma^\infty(A(G))$ of the vector bundle $A(G) = \ker(\mathrm{d}t)|_{u(M)}$ over $u(M) \approx M$.

Proof. For the first part, first note that if X and Y are tangent to t-fibres, then so is $[X, Y]$; furthermore, note that for any $g\colon x \to y$, the map $L_g\colon G^x \to G^y$ is a diffeomorphism, and that left-invariance of X just means that $X|_{G^x}$ is L_g-related to $X|_{G^y}$, and similarly for Y. This implies that $[X, Y]|_{G^x}$ is L_g-related to $[X, Y]|_{G^y}$, i.e. $\mathrm{d}(L_g)_h([X, Y]_h) = [X, Y]_{gh}$ for all $(g, h) \in G * G$.

For the second part, the isomorphism reads

$$\mathfrak{X}_L(G) \to \Gamma^\infty(A(G)), \quad X \mapsto X|_{u(M)},$$

and its inverse is given by $\alpha \mapsto X^\alpha$, where $X^\alpha_g = \mathrm{d}(L_g)_{1_{s(g)}}(\alpha_{1_{s(g)}})$. Smoothness of X^α follows from a straightforward verification that there holds $X^\alpha = \mathrm{d}m \circ \tau_\alpha$, where $\tau_\alpha\colon G \to T(G * G)$ is given by $\tau_\alpha(g) = (0_g, \alpha_{1_{s(g)}})$. ■

This lemma now allows us to transfer the Lie bracket from $\mathfrak{X}_L(G)$ to $\Gamma^\infty(A(G))$, by defining it as the map $\Gamma^\infty(A(G)) \times \Gamma^\infty(A(G)) \to \Gamma^\infty(A(G))$, $(\alpha, \beta) \mapsto [\alpha, \beta]$, where

$$[\alpha, \beta]_x := [X^\alpha, X^\beta]_{1_x} \quad \text{for any } x \in M.$$

This map obviously satisfies all the axioms of a Lie bracket, since the Lie bracket on $\mathfrak{X}(G)$ does.

We establish more of its properties by observing that we may also define the *anchor* of $A(G)$, which is a vector bundle morphism

$$\rho\colon A(G) \to TM, \quad \rho = \mathrm{d}s|_{A(G)},$$

i.e. $\rho(\alpha) = \mathrm{d}s_{1_x}(\alpha)$ for any $\alpha \in \ker \mathrm{d}t_{1_x}$.

Proposition 3.9. On any Lie groupoid $G \rightrightarrows M$, the anchor $\rho\colon A(G) \to TM$ satisfies:

(i) $\rho[\alpha, \beta] = [\rho\alpha, \rho\beta]$ for all $\alpha, \beta \in \Gamma^\infty(A(G))$.

(ii) *Leibniz rule:* $[\alpha, f\beta] = f[\alpha, \beta] + \rho(\alpha)(f)\beta$ for all $\alpha, \beta \in \Gamma^\infty(A(G))$ and $f \in C^\infty(M)$.

Proof. We first prove (i). The left-invariant vector field X^α is s-related to $\rho\alpha$, since

$$\mathrm{d}s_g(X^\alpha_g) = \mathrm{d}(s \circ L_g)_{1_{s(g)}}(\alpha_{1_{s(g)}}) = \rho(\alpha_{1_{s(g)}}),$$

and similarly for β, hence also $[\rho\alpha, \rho\beta]$ is s-related to $[X^\alpha, X^\beta]$, that is,

$$\mathrm{d}s_g([X^\alpha, X^\beta]_g) = [\rho\alpha, \rho\beta]_{s(g)}.$$

Now take $g = 1_x$.

To prove (ii), note that $X^{f\alpha}_g = \mathrm{d}(L_g)_{1_{s(g)}}(f(s(g))\alpha_{1_{s(g)}}) = \big((f \circ s)X^\alpha\big)|_g$, so there holds $X^{f\alpha} = (f \circ s)X^\alpha$. Now we take the left-invariant vector field which corresponds to $[\alpha, f\beta]$:

$$\begin{aligned} X^{[\alpha, f\beta]} &= [X^\alpha, X^{f\beta}] = [X^\alpha, (f \circ s)X^\beta] \\ &= (f \circ s)[X^\alpha, X^\beta] + X^\alpha(f \circ s)X^\beta \\ &= X^{(f\circ s)[\alpha,\beta]} + X^{\rho(\alpha)(f)\beta} \\ &= X^{(f\circ s)[\alpha,\beta]+\rho(\alpha)(f)\beta}, \end{aligned}$$

where we have used the usual Leibniz rule in the second line and the equality $X^\alpha(f \circ s) = \mathrm{d}s(X^\alpha)(f)$ in the third. By previous lemma, finishes the proof. ■

Remark 3.10. We thus see that if $G \rightrightarrows M$ is a Lie groupoid, the triple $(A(G), [\cdot,\cdot], \rho)$ as defined above, is a Lie algebroid. We call it the *Lie algebroid* of $G \rightrightarrows M$.

Example 3.11.

(i) *Lie algebroid of pair groupoid.* The Lie algebroid $A(M \times M)$ of the groupoid $M \times M \rightrightarrows M$ can be identified with the tangent bundle $TM \to M$. To see this, note that $\ker \mathrm{d}t_{(x,x)} = 0_x \oplus T_x M$, thus $\ker \mathrm{d}t|_{u(M)} = 0 \oplus TM$, and $\rho = \mathrm{d}s|_{0\oplus TM} = \mathrm{d}(\mathrm{pr}_2)|_{0\oplus TM}$, so this is just the projection $0 \oplus TM \to TM$, which we identify with id_{TM}. That the bracket can be identifed with the Lie bracket on TM is also a triviality.

(ii) *Action algebroid.* If a Lie group G acts on M from the right, we get a map $a\colon \mathfrak{g} \to \mathfrak{X}(M)$,

$$a(v)(x) = \frac{d}{d\lambda}\bigg|_{\lambda=0} (x \cdot \exp(\lambda v))$$

which is a homomorphism of Lie algebras. The *action Lie algebroid* is defined to be the trivial bundle $M \times \mathfrak{g} \to M$, hence $\Gamma^\infty(M \times \mathfrak{g}) = C^\infty(M, \mathfrak{g})$. The anchor of this algebroid is defined to be $\rho(x, v) = a(v)(x)$ and the Lie bracket $[\cdot,\cdot]$ as

$$[f, g](x) = [f(x), g(x)]_{\mathfrak{g}} + \mathrm{d}g_x(a(f(x))) - \mathrm{d}f_x(a(g(x))),$$

i.e. the unique Lie bracket on $\Gamma^\infty(M \times \mathfrak{g})$ satisfying Leibniz rule and the equality $[c_v, c_w] = c_{[v,w]_{\mathfrak{g}}}$, where c_v denotes the constant map into $v \in \mathfrak{g}$.

Let us show that this is in fact the Lie algebroid of $M \rtimes G \rightrightarrows M$. First off, notice that $\ker \mathrm{d}t_{(x,e)} = 0_x \oplus T_e G = 0_x \times \mathfrak{g}$, thus $A(M \rtimes G) = \ker \mathrm{d}t|_{u(M)} = M \times \mathfrak{g}$. The anchor reads

$$\mathrm{d}s|_{M\times\mathfrak{g}}(0_x, v) = \frac{d}{d\lambda}\bigg|_{\lambda=0} t(x, \exp(\lambda v)) = a(v, x),$$

and the Lie bracket on $A(M \rtimes G)$ on constant maps reads

$$[c_v, c_w]|_{(x,e)} = [X^{c_v}, X^{c_w}]|_{(x,e)} = X^{c_{[v,w]_{\mathfrak{g}}}}|_{(x,e)} = c_{[v,w]_{\mathfrak{g}}}|_{(x,e)},$$

where we have used the definition $X^{c_v}|_{(x,g)} = \mathrm{d}(L_{(x,g)})_{(xg,e)}(v)$ of a left-invariant vector field, and used the result regarding relatedness and Lie brackets. Thus the bracket on $A(M \rtimes G)$ coincides with the bracket on the action algebroid (by uniqueness).

4 Symplectic groupoids

To introduce the notion of a symplectic groupoid, we first consider what it means for a differential form to be compatible with groupoid structure.

4.1 Multiplicative forms on Lie groupoids

To motivate the definition of multiplicative forms on a Lie groupoid $G \rightrightarrows M$, let us first consider a 0-form $f \in C^\infty(G)$. We say that the function f is *multiplicative*, if there holds

$$f(gh) = f(g) + f(h)$$

for any composable arrows $g, h \in G$. Put another way, $f\colon G \to (\mathbb{R}, +)$ is a morphism of groupoids; the reason behind working with $(\mathbb{R}, +)$ instead of $(\mathbb{R}\backslash\{0\}, \cdot)$ is in that we are allowing f to have zeros. Denoting by $\mathrm{pr}_1, \mathrm{pr}_2\colon G * G \to G$ the restriction of projections $G \times G \to G$, the above equality may be written as

$$m^* f = f \circ m = f \circ \mathrm{pr}_1 + f \circ \mathrm{pr}_2 = \mathrm{pr}_1^* f + \mathrm{pr}_2^* f.$$

This condition now makes sense for differential forms of arbitrary degree.

Definition 4.1. A *multiplicative k-form* on a Lie groupoid $G \rightrightarrows M$ is a differential k-form $\omega \in \Omega^k(G)$, such that

$$m^*\omega = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega.$$

Lemma 4.2. Let $k \geq 1$. For any multiplicative k-form $\omega \in \Omega^k(G)$ on a Lie groupoid $G \rightrightarrows M$, there holds

$$\begin{aligned} \mathrm{d}(L_g)_h{}^*\big(\omega|_{\ker \mathrm{d}t_{gh}}\big) &= \omega|_{\ker \mathrm{d}t_h}, \\ \mathrm{d}(R_h)_g{}^*\big(\omega|_{\ker \mathrm{d}s_{gh}}\big) &= \omega|_{\ker \mathrm{d}s_g}, \end{aligned}$$

for any composable arrows $(g, h) \in G * G$.

Remark 4.3. These equalities are saying in a precise way that there holds "$(L_g)^*\omega = \omega$" and "$(R_h)^*\omega = \omega$" wherever the pullbacks make sense – we have to keep in mind that the diffeomorphisms $L_g\colon G^{s(g)} \to G^{t(g)}$ and $R_h\colon G_{t(h)} \to G_{s(h)}$ are not globally defined maps.

Proof. We only prove the second identity; the proof of the first one is similar. If $v \in T_gG$, we have $\mathrm{d}(R_h)_g(v) = \mathrm{d}m_{(g,h)}(v, 0_h)$. On the other hand, multiplicativity of ω implies

$$(m^*\omega)_{(g,h)}((v_1, 0_h), \dots, (v_k, 0_h)) = \omega_g(v_1, \dots, v_k),$$

for all $v_i \in T_gG$, so we must have

$$\omega_{gh}(\mathrm{d}(R_h)_g(v_1), \dots, \mathrm{d}(R_h)_g(v_k)) = \omega_g(v_1, \dots, v_k),$$

To conclude, just note that since $R_h\colon G_{t(h)} \to G_{s(h)}$ is a diffeomorphism, the induced map of tangent spaces $\mathrm{d}(R_h)_g\colon \ker \mathrm{d}s_g \to \ker \mathrm{d}s_{gh}$ is an isomorphism. ∎

An important point of view for multiplicativity of differential forms on G is the following cohomological condition. We have the maps

$$\begin{aligned} \delta^1\colon \Omega^k(G) \to \Omega^k(G * G), &\quad \delta^1\omega = \mathrm{pr}_1^*\omega - m^*\omega + \mathrm{pr}_2^*\omega, \\ \delta^0\colon \Omega^k(M) \to \Omega^k(G), &\quad \delta^0\alpha = s^*\alpha - t^*\alpha. \end{aligned}$$

Observe that $\omega \in \Omega^k(G)$ is multiplicative if and only if it is a *cocycle*, i.e. $\delta^1\omega = 0$. Furthermore, $\delta^1 \circ \delta^0 = 0$ holds, since:

$$\begin{aligned}\delta^1\delta^0\alpha = \delta^1(s^*\alpha - t^*\alpha) &= \mathrm{pr}_1^*(s^*\alpha) - m^*(s^*\alpha) + \mathrm{pr}_2^*(s^*\alpha)\\ &\quad - \mathrm{pr}_1^*(t^*\alpha) + m^*(t^*\alpha) - \mathrm{pr}_2^*(t^*\alpha),\end{aligned}$$

and now observe that the first and last term cancel out since $t \circ \mathrm{pr}_2 = s \circ \mathrm{pr}_1$, second and third term cancel out since $s \circ m = s \circ \mathrm{pr}_2$, and the remaining two terms cancel out since $t \circ \mathrm{pr}_1 = t \circ m$. This shows that the *coboundaries* $\delta^0\alpha$, for any $\alpha \in \Omega^k(M)$, are multiplicative forms on G.

4.2 Multiplicative symplectic structures

Definition 4.4. A *symplectic groupoid* is a Lie groupoid $\Sigma \rightrightarrows M$ together with a multiplicative symplectic form $\Omega \in \Omega^2(\Sigma)$.

In what follows, we will use the letter Σ to distinguish symplectic groupoids from ordinary Lie groupoids, and denote by Ω the multiplicative symplectic form on Σ.

Example 4.5.

(i) *Symplectic manifolds induce a symplectic pair groupoid.* If (M, ω) is a symplectic manifold, then the pair groupoid $\Sigma = M \times M \rightrightarrows M$ is a symplectic groupoid with the multiplicative symplectic form given as

$$\Omega = \mathrm{pr}_1^*\omega - \mathrm{pr}_2^*\omega = -\delta^0\omega.$$

This form is indeed closed since exterior derivative commutes with pullbacks, and it is a symplectic form since

$$\Omega^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} (\mathrm{pr}_1^*\omega)^k \wedge (-\mathrm{pr}_2^*\omega)^{2n-k} = (-1)^n \binom{2n}{n} (\mathrm{pr}_1^*\omega)^n \wedge (\mathrm{pr}_2^*\omega)^n$$

vanishes nowhere.

(ii) *Canonical symplectic forms on cotangent bundles are multiplicative.* If M is a smooth manifold, the cotangent bundle T^*M admits a structure of a symplectic groupoid. To first see that it admits a structure of a Lie groupoid, declare both the source and the target map as the canonical projection $p\colon T^*M \to M$, and let multiplication $m\colon T^*M \oplus T^*M \to T^*M$ in this groupoid be given as addition $m(\alpha, \beta) = \alpha + \beta$ (on the fibres), with the obvious inversion and unit; the structure maps are then clearly smooth.

To see that the canonical symplectic structure on T^*M is multiplicative, consider the tautological 1-form $\tau \in \Omega^1(T^*M)$. We claim that it is multiplicative, and to show this, we first let $\mathrm{pr}_1, \mathrm{pr}_2\colon T^*M \oplus T^*M \to T^*M$ denote the projections. Let $(v, w) \in T_{(\alpha,\beta)}(T^*M \oplus T^*M)$ where $p(\alpha) = p(\beta) = x \in M$, and by (10), we must have $\mathrm{d}p_\alpha(v) = \mathrm{d}p_\beta(w) =: u \in T_xM$. Since the tautological 1-form reads $\tau_\xi = \xi \circ \mathrm{d}p_\xi$ for any $\xi \in T^*M$, we get

$$\begin{aligned}\tau_{\alpha+\beta}(\mathrm{d}m_{(\alpha,\beta)}(v, w)) &= (\alpha + \beta)(\mathrm{d}(p \circ m)_{(\alpha,\beta)}(v, w)) = (\alpha + \beta)(\mathrm{d}p_\alpha(v)) = (\alpha + \beta)(u)\\ &= \alpha(u) + \beta(u) = \alpha(\mathrm{d}p_\alpha(v)) + \beta(\mathrm{d}p_\beta(w))\\ &= \tau_\alpha(\mathrm{d}(\mathrm{pr}_1)_{(\alpha,\beta)}(v, w)) + \tau_\beta(\mathrm{d}(\mathrm{pr}_1)_{(\alpha,\beta)}(v, w)),\end{aligned}$$

proving $m^*\tau = \mathrm{pr}_1^*\tau + \mathrm{pr}_2^*\tau$. It is clear from this that the canonical symplectic structure $\omega = -\,\mathrm{d}\tau$ is multiplicative with respect to the given Lie groupoid structure on T^*M.

(iii) *Coadjoint action groupoid is symplectic.* Let G be a Lie group and let $\Sigma = G \times \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$ be the action groupoid of the left action $g \cdot \xi = \mathrm{Ad}^*_g \xi$ of G on $\mathfrak{g}^*$, where $(\mathrm{Ad}^*_g \xi)(v) = \xi(\mathrm{Ad}_{g^{-1}}(v))$ for any $v \in \mathfrak{g}$. This is called the *coadjoint action* of G on $\mathfrak{g}^*$.

Note that we have the isomorphism $T^*G \cong G \times \mathfrak{g}^*$ of vector bundles, given by

$$\ell \colon T^*G \to G \times \mathfrak{g}^* = \Sigma, \quad T^*_g G \ni \alpha \mapsto (g, \alpha \circ \mathrm{d}(L_g)_e),$$

and now we consider the 2-form

$$\Omega := -\ell_* \omega = \mathrm{d}(\ell_* \tau)$$

on Σ, where $\omega = -\,\mathrm{d}\tau$ denotes the canonical symplectic form on T^*G, τ is the tautological 1-form on T^*G, and we write $\ell_* = (\ell^{-1})^*$. To show that $\ell_*\tau$ is multiplicative, we note it is pointwise given as the map $(\ell_*\tau)_{(g,\xi)} \colon T_{(g,\xi)}(G \times \mathfrak{g}^*) \to \mathbb{R}$,

$$(\ell_*\tau)_{(g,\xi)}(v, \eta) = \xi(\mathrm{d}(L_{g^{-1}})_g(v)) \quad \text{for any } v \in T_g G \text{ and } \eta \in \mathfrak{g}^* \cong T_\xi \mathfrak{g}^*.$$

To see this, define the right hand side as $\theta_{(g,\xi)}(v,\eta)$, so that $\theta \in \Omega^1(\Sigma)$, and compute at any $\alpha \in T^*_g G$, for any $u \in T_\alpha(T^*G)$:

$$\begin{aligned}(\ell^*\theta)_\alpha(u) &= \theta_{\ell(\alpha)}(\mathrm{d}\ell_\alpha(u)) = (\alpha \circ \mathrm{d}(L_g)_e)(\mathrm{d}(L_{g^{-1}})_g(\mathrm{d}(\mathrm{pr}_1 \circ \ell)_\alpha(u))) \\ &= \alpha(\mathrm{d}p_\alpha(u)) = \tau_\alpha(u)\end{aligned}$$

where $\mathrm{pr}_1 \colon G \times \mathfrak{g}^*$ is the projection onto the first factor, and we have used $\mathrm{pr}_1 \circ \ell = p$. This shows $\ell^*\theta = \tau$, hence $\ell_*\tau = \theta$, as claimed.

So we must show that θ is multiplicative. To that end, we identify $\Sigma * \Sigma$ with $G \times G \times \mathfrak{g}^*$, using the bijection $((g, \mathrm{Ad}^*_h \xi), (h, \xi)) \leftrightarrow (g, h, \xi)$. That way, our partial multiplication reads $m(g,h,\xi) = (gh, \xi)$ and the projection maps $\mathrm{pr}_1, \mathrm{pr}_2 \colon G \times G \times \mathfrak{g}^* \to G \times \mathfrak{g}^*$ read

$$\mathrm{pr}_1(g,h,\xi) = (g, \mathrm{Ad}^*_h \xi), \quad \mathrm{pr}_2(g,h,\xi) = (h,\xi).$$

Lastly, we compute

$$\begin{aligned}(m^*\theta)_{(g,h,\xi)}(v,w,\eta) &= \xi\big(\mathrm{d}(L_{(gh)^{-1}})_{gh}(\mathrm{d}(R_h)_g(v) + \mathrm{d}(L_g)_h(w))\big) \\ &= \xi(\mathrm{Ad}_{h^{-1}}(\mathrm{d}(L_{g^{-1}})_g(v)) + \mathrm{d}(L_{h^{-1}})_h(w))\end{aligned}$$

where we have computed $L_{(gh)^{-1}} \circ R_h = C_{h^{-1}} \circ L_{g^{-1}}$. On the other hand, we have

$$\begin{aligned}(\mathrm{pr}_1^*\theta)_{(g,h,\xi)}(v,w,\eta) &= \theta_{(g,\mathrm{Ad}^*_h\xi)}(\mathrm{d}(\mathrm{pr}_1)_{(g,h,\xi)}(v,w,\eta)) = (\mathrm{Ad}^*_h\xi)(\mathrm{d}(L_{g^{-1}})_g(v)) \\ &= \xi(\mathrm{Ad}_{h^{-1}}(\mathrm{d}(L_{g^{-1}})_g(v)))\end{aligned}$$

and $(\mathrm{pr}_2^*\theta)_{(g,h,\xi)}(v,w,\eta) = \theta_{(h,\xi)}(\mathrm{d}(\mathrm{pr}_2)_{(g,h,\xi)}(v,w,\eta)) = \xi(\mathrm{d}(L_{h^{-1}})_h(w))$. Hence

$$m^*\theta = \mathrm{pr}_1^*\theta + \mathrm{pr}_2^*\theta,$$

so θ is multiplicative, as claimed.

4.3 Geometric interpretations of multiplicativity

We now come to the very important geometric consequences of multiplicativity of a symplectic form. In what's to come, we will also see that existence of a multiplicative symplectic structure imposes dimensional restrictions on a Lie groupoid.

Lemma 4.6. A differential 2-form $\Omega \in \Omega^2(G)$ on a Lie groupoid G is multiplicative if and only if the graph

$$\Gamma(m) = \{(g, h, gh) \mid (g, h) \in G * G\} \subset \Sigma \times \Sigma \times \Sigma$$

of the partial multiplication map m, is isotropic in $(G, \Omega) \times (G, \Omega) \times (G, -\Omega)$, i.e. the form $\Omega \oplus \Omega \oplus (-\Omega)$ restricted to $\Gamma(m)$ vanishes.

Proof. Defining the map $\phi\colon G * G \to G \times G \times G$ as

$$\phi(g, h) = (g, h, gh) = (\mathrm{pr}_1(g, h), \mathrm{pr}_2(g, h), m(g, h)),$$

we see that $\Gamma(m)$ is the image of ϕ, and we clearly have

$$\phi^*(\mathrm{pr}_1^*\Omega + \mathrm{pr}_2^*\Omega - \mathrm{pr}_3^*\Omega) = \mathrm{pr}_1^*\Omega + \mathrm{pr}_2^*\Omega - m^*\Omega,$$

where we note that the projections $\mathrm{pr}_i\colon G \times G \times G \to G$ on the left side differ from those on the right side, and $m = \mathrm{pr}_3 \circ \phi$. Since ϕ is an immersion, we must have

$$\mathrm{pr}_1^*\Omega + \mathrm{pr}_2^*\Omega - \mathrm{pr}_3^*\Omega = 0 \text{ if and only if } m \text{ is multiplicative.} \qquad \blacksquare$$

Proposition 4.7. A symplectic form $\Omega \in \Omega^2(G)$ on $G \rightrightarrows M$ is multiplicative if and only if the graph

$$\Gamma(m) \subset G \times G \times G$$

of partial multiplication map, is a Lagrangian submanifold. Moreover, any symplectic groupoid (Σ, Ω) has the following properties:

(i) t-fibres and s-fibres are symplectic orthogonal, i.e.

$$(\ker \mathrm{d}t)^\Omega = \ker \mathrm{d}s.$$

(ii) The unit map $u\colon M \to \Sigma$ is a Lagrangian embedding, i.e.

$$(TM)^\Omega = TM.$$

(iii) Inversion $\mathrm{inv}\colon \Sigma \to \Sigma$ is an antisymplectomorphism, i.e.

$$\mathrm{inv}^*\Omega = -\Omega.$$

In particular, $\dim \Sigma = 2 \dim M$.

Proof. If the graph $\Gamma(m)$ is Lagrangian, the last lemma shows that Ω is multiplicative. Conversely, suppose (Σ, Ω) is a symplectic groupoid. For the first part, it is enough to show that $\dim \Sigma = 2 \dim M$, since it then follows that $\dim \Gamma(m) = 2 \dim \Sigma - \dim M = 3 \dim M = \frac{1}{2} \dim(\Sigma \times \Sigma \times \Sigma)$, so again by the last lemma, $\Gamma(m)$ is Lagrangian.

We first show that $u^*\Omega = 0$; pulling back the multiplicativity condition along the map $\bar{u}\colon M \to \Sigma * \Sigma, \bar{u}(x) = (u(x), u(x))$, yields

$$0 = \bar{u}^*\delta\Omega = \bar{u}^*\mathrm{pr}_1^*\Omega - \bar{u}^*m^*\Omega + \bar{u}^*\mathrm{pr}_2^*\Omega = u^*\Omega - u^*\Omega + u^*\Omega = u^*\Omega,$$

where we've observed $\mathrm{pr}_2 \circ \bar{u} = \mathrm{pr}_1 \circ \bar{u} = m \circ \bar{u} = u$. This means that $u(M)$ is an isotropic submanifold in Σ.

Secondly, we show that inversion is an antisymplectomorphism; defining the map $\Delta\colon \Sigma \to \Sigma * \Sigma, \Delta(g) = (g, g^{-1})$, noting $m \circ \Delta = u$, and using the equation $u^*\Omega = 0$, yields

$$0 = u^*\Omega = \Delta^* m^* \Omega = \Delta^*(\mathrm{pr}_1^*\Omega + \mathrm{pr}_2^*\Omega) = \Omega + \mathrm{inv}^*\Omega,$$

where we've observed $\mathrm{pr}_1 \circ \Delta = \mathrm{id}_G$ and $\mathrm{pr}_2 \circ \Delta = \mathrm{inv}$. This proves (iii), and now note that the fixed point set of any involutive antisymplectomorphism is a coisotropic submanifold (see lemma after the proof), and $u(M) \subset \mathrm{Fix}(\mathrm{inv})$, hence $u(M)$ is coisotropic. Together with the previous paragraph, this shows that $u(M)$ is a Lagrangian submanifold in Σ, which shows (ii) and proves the dimension correspondence $\dim \Sigma = 2 \dim M$.

Finally, to prove (i), by the dimensionality, it's enough to show $\Omega(v, w) = 0$ holds for any $v \in \ker \mathrm{d}s_g$ and $w \in \ker \mathrm{d}t_g$, since this implies $\ker \mathrm{d}t \subset (\ker \mathrm{d}s)^\Omega$ and $\ker \mathrm{d}s \subset (\ker \mathrm{d}t)^\Omega$ and both subbundles $\ker \mathrm{d}s$ and $\ker \mathrm{d}t$ of $T\Sigma$ have rank $\dim M$ since s and t are submersions. To do that using multiplicativity of Ω, we would like to rewrite v, w as pushforwards by m. Note that $v \in \ker \mathrm{d}s_g$ implies $(v, 0_{1_{s(g)}}) \in T_{(g, 1_{s(g)})}(\Sigma * \Sigma)$ by formula (10), and clearly

$$v = \mathrm{d}(R_{1_{s(g)}})_g(v) = \mathrm{d}m_{(g,1_{s(g)})}(v, 0_{1_{s(g)}}).$$

On the other hand, note that $w \in \ker \mathrm{d}t_g$ implies $(0_g, w_0) \in T_{(g, 1_{s(g)})}(\Sigma * \Sigma)$ where we have written $w_0 = \mathrm{d}(L_{g^{-1}})_g(w)$, and similarly to above,

$$w = \mathrm{d}m_{(g,1_{s(g)})}(0_g, w_0).$$

Hence

$$\begin{aligned}\Omega(v, w) &= \Omega(\mathrm{d}m(v, 0_{1_{s(g)}}), \mathrm{d}m(0_g, w_0)) \\ &= \Omega(\mathrm{d}(\mathrm{pr}_1)(v, 0_{1_{s(g)}}), \mathrm{d}(\mathrm{pr}_1)(0_g, w_0)) + \Omega(\mathrm{d}(\mathrm{pr}_2)(v, 0_{1_{s(g)}}), \mathrm{d}(\mathrm{pr}_2)(0_g, w_0)) = 0. \qquad \blacksquare\end{aligned}$$

Lemma 4.8. Let (M, ω) be a symplectic manifold. If $\phi\colon M \to M$ is an involutive antisymplectomorphism, i.e. $\phi^2 = \mathrm{id}_M$ and $\phi^*\omega = -\omega$, then its fixed point set is a coisotropic submanifold.

Proof. Denote the fixed point set of ϕ by $F = \{x \in M \mid \phi(x) = x\}$. This is a submanifold in M, and at any point $x \in F$, its tangent space reads $T_xF = \{v \in T_xM|_F \mid \mathrm{d}\phi_x(v) = v\}$. Since ϕ is involutive, we have $(\mathrm{d}\phi_x)^2 = \mathrm{id}_{T_xM}$, so T_xM decomposes to (± 1)-eigenspaces, denoted $(T_xM)^\pm$; observe that $(T_xM)^+ = T_xF$. On the other hand, we also have $(\mathrm{d}\phi_x^*)^2 = \mathrm{id}_{T_x^*M}$, so that T_x^*M also decomposes to (± 1)-eigenspaces, denoted $(T_x^*M)^\pm$; now observe that

$$(T_x^*M)^- = \{\xi \in T_x^*M \mid \xi \circ \mathrm{d}\phi_x = -\xi\} = \{\xi \in T_x^*M \mid \xi(v) = 0 \text{ for all } v \in T_xF\} = (T_xF)^\circ$$

Take the nondegenerate Poisson structure $\pi^\sharp = (\omega^\flat)^{-1}$, associated to ω. Since ϕ is an antisymplectomorphism, we have by Proposition 1.12 that

$$-\pi_x^\sharp = \mathrm{d}\phi_x \circ \pi_x^\sharp \circ (\mathrm{d}\phi_x)^*,$$

which implies that for any $\alpha \in (T_x^*M)^-$, we have $\pi_x^\sharp(\alpha) \in (T_xM)^+$, which is equivalent to $(\omega^\flat)^{-1}((T_xF)^\circ) \subset T_xF$, and furthermore to $(T_xF)^\omega \subset T_xF$, which is what we wanted. $\blacksquare$

4.4 Interplay of symplectic groupoids and Poisson manifolds

We now have all the tools to prove the most insightful result of this exposition, the proof of which, being straightforward albeit abstract, shows that symplectic groupoids have an important place in Poisson geometry.

Theorem 4.9. Let $\Sigma \rightrightarrows M$ be a symplectic groupoid. There is a unique Poisson structure π on M, such that $t\colon (\Sigma, \Omega) \to (M, \pi)$ is a Poisson map. Moreover,

(i) $t\colon \Sigma \to M$ is a complete symplectic realization, i.e. for any complete Hamiltonian vector field $X_f \in \mathfrak{X}^1(M)$, the Hamiltonian vector field $X_{f\circ t} \in \mathfrak{X}^1(\Sigma)$ is also complete.

(ii) There is a canonical isomorphism of Lie algebroids

$$\sigma_\Omega\colon A(\Sigma) \to T^*M, \quad \sigma_\Omega(\alpha) = -u^*(\Omega^\flat(\alpha)).$$

In particular, $\pi^\sharp = \rho \circ \sigma_\Omega^{-1}$, where ρ denotes the anchor of $A(\Sigma)$.

(iii) The infinitesimal action of the symplectic realization $t\colon \Sigma \to M$ is given by

$$\mathfrak{a}\colon \Omega^1(M) \to \mathfrak{X}^1(\Sigma), \quad \mathfrak{a}(\sigma_\Omega(\alpha)) = X^\alpha,$$

where X^α is the left-invariant vector field that corresponds to $\alpha \in \Gamma^\infty(A(\Sigma))$.

Remark 4.10. Among the properties of $t\colon (\Sigma, \Omega) \to (M, \pi)$ is also the fact that the symplectic leaves of (M, π) are the connected components of orbits of Σ, which we will not prove.

Proof. Let us show that the target map t pushes forward the inverse $(\Omega^\flat)^{-1}$ of the isomorphism $\Omega^\flat\colon T\Sigma \to T^*\Sigma$, to a bivector π on M; recall $\Omega^\flat$ is given by $v \mapsto \Omega(v, \cdot)$.

We define π in the following way. Fix $x \in M$, and let $g \in G^x$ be an arbitrary arrow in G with target x; then define the map $(\pi_x^g)^\sharp\colon T_x^*M \to T_xM$ using the following composition:

$$\begin{tikzcd}
T_g^*\Sigma \arrow[r, "(\Omega_g^\flat)^{-1}"] & T_g\Sigma \arrow[d, "\mathrm{d}t_g"] \\
T_x^*M \arrow[u, "(\mathrm{d}t_g)^*"] \arrow[r, "(\pi_x^g)^\sharp"'] & T_xM
\end{tikzcd}$$

This map is independent of the choice of $g \in G^x$. Indeed, note that if $h \in G^x$ is another arrow, then we must have $h = gk$ for some $k \in G$, namely $k = g^{-1}h$, and now

$$\mathrm{d}t_h = \mathrm{d}t_{gk} = \mathrm{d}(t \circ R_{k^{-1}})_{gk} = \mathrm{d}t_g \circ \mathrm{d}(R_{k^{-1}})_h,$$

so that $(\mathrm{d}t_h)^* = \mathrm{d}(R_{k^{-1}})_h{}^* \circ (\mathrm{d}t_g)^*$, and we get

$$(\pi_x^h)^\sharp = \mathrm{d}t_g \circ \mathrm{d}(R_{k^{-1}})_h \circ (\Omega_h^\flat)^{-1} \circ \mathrm{d}(R_{k^{-1}})_h{}^* \circ (\mathrm{d}t_g)^*,$$

but $\mathrm{d}(R_{k^{-1}})_h \circ (\Omega_h^\flat)^{-1} \circ \mathrm{d}(R_{k^{-1}})_h{}^* = (\Omega_g^\flat)^{-1}$ holds by multiplicativity of Ω. Indeed, note that this is a composition of isomorphisms, since $\mathrm{d}(R_{k^{-1}})_h\colon \ker \mathrm{d}s_h \to \ker \mathrm{d}s_g$ is an isomorphism; thus it suffices to check the inverse of the wanted equality, which is

$$\mathrm{d}(R_k)_h{}^* \circ \Omega_h^\flat \circ \mathrm{d}(R_k)_h = \Omega_g^\flat,$$

and this is precisely the second equality in Lemma 4.2.

To show that the obtained bivector field π is indeed a Poisson structure, we just note that the Libermann's theorem now applies locally, because $(\ker \mathrm{d}t)^{\Omega} = \ker \mathrm{d}s$ is clearly an integrable distribution; hence by uniqueness in Libermann's theorem, π is necessarily smooth and Poisson, and the target map is its symplectic realization.

Next, we have to check that σ_Ω is a isomorphism of Lie algebroids; denote the Lie algebroid of Σ by $(A, [\cdot,\cdot]_A, \rho)$. First, we show that σ_ρ is pointwise an isomorphism of vector spaces. To show this, note that[§] $T_xM \subset T_{u(x)}\Sigma$ is a Lagrangian subspace by Proposition 4.7, so it is precisely the kernel of the map

$$T_{u(x)}\Sigma \to T_x^*M, \quad v \mapsto -\Omega(v,\cdot)|_{T_xM} = -\Omega^\flat(v)|_{T_xM}.$$

Because $T_{u(x)}\Sigma = T_xM \oplus A_x$ and Ω is nondegenerate, this map restricts to an isomorphism $A_x \to T_x^*M$. Up to the identification $M \approx u(M)$ with respect to the embedding u, this is precisely the map $\sigma_\Omega|_x$, hence σ_Ω is an isomorphism of vector bundles (any smooth morphism of vector bundles that is fibrewise an isomorphism, is an isomorphism of vector bundles).

To show that σ_Ω is a morphism of Lie algebroids, it is enough to show that it preserves the Lie brackets,[¶] and to this end we inspect the infinitesimal action of the obtained symplectic realization; its definition in our case reads

$$\mathfrak{a}\colon \Omega^1(M) \to \mathfrak{X}^1(\Sigma), \quad \Omega^\flat \circ \mathfrak{a} = t^*.$$

As in property (iii) of the theorem, we claim $\mathfrak{a}(\sigma_\Omega(\alpha)) = X^\alpha$ for any $\alpha \in \Gamma^\infty(A)$, so we must prove $\Omega^\flat(X^\alpha) = t^*(\sigma_\Omega(\alpha))$, or equivalently (since inv is an antisymplectomorphism),

$$\Omega^\flat(X^\alpha) = -s^*(\sigma_\Omega(\alpha)), \tag{11}$$

where X^α denotes the left-invariant vector fields associated to α, i.e.

$$X^\alpha|_g = \mathrm{d}(L_g)_{1_{s(g)}}(\alpha_{1_{s(g)}}) = \mathrm{d}m_{(g,1_{s(g)})}(0_g, \alpha_{1_{s(g)}}).$$

To evaluate both sides on a vector $v \in T_g\Sigma$, note that we can write

$$v = \mathrm{d}m_{(g,1_{s(g)})}(v, \mathrm{d}u_{s(g)}\,\mathrm{d}s_g(v))$$

since $L_g \circ u \circ s$ is a constant map into g. Hence left hand side of (11) is

$$\begin{aligned}\Omega_g(X^\alpha|_g, v) &= \Omega_g\big(\mathrm{d}m_{(g,1_{s(g)})}(0_g, \alpha_{1_{s(g)}}), \mathrm{d}m_{(g,1_{s(g)})}(v, \mathrm{d}u_{s(g)}\,\mathrm{d}s_g(v))\big)\\ &= \cancel{\Omega_g(0_g, v)} + \Omega_{1_{s(g)}}(\alpha_{1_{s(g)}}, \mathrm{d}u_{s(g)}\,\mathrm{d}s_g(v)),\end{aligned}$$

and the right hand side, with indices left out, is

$$-s^*(\sigma_\Omega(\alpha))(v) = -\sigma_\Omega(\alpha)(\mathrm{d}s(v)) = u^*(\Omega^\flat(\alpha))(\mathrm{d}s(v)) = \Omega(\alpha, \mathrm{d}u(\mathrm{d}s(v))),$$

hence our claim holds. Since the infinitesimal action preserves brackets by Proposition 2.9, and the bracket on A comes from the bracket of left-invariant vector fields, σ_Ω also preserves Lie brackets.

[§] We are identifying $M \approx u(M)$.

[¶] An isomorphism of vector bundles $A \to B$ that preserves Lie brackets must preserve anchors as well; this follows from the fact that if $[\cdot,\cdot]$ is a Lie bracket on the sections of a vector bundle A, there is at most one anchor on A making it into a Lie algebroid, which is a simple consequence of the Leibniz rule.

To conclude, we have to prove completeness of our symplectic realization; again, it is equivalent to prove that the symplectic realization $s\colon (\Sigma, -\Omega) \to (M, \pi)$ is complete. So, suppose $X_f \in \mathfrak{X}(M)$ is a complete Hamiltonian vector field of $f \in C^\infty(M)$. We define $\alpha = \sigma_\Omega^{-1}(\mathrm{d}f)$, and by (11), we have

$$\Omega^\flat(X^\alpha) = -\,\mathrm{d}(f \circ s),$$

implying that X^α is the Hamiltonian vector field of $f \circ s$ on $(\Sigma, -\Omega)$. Since

$$X_f = \pi^\sharp(\mathrm{d}f) = \rho(\sigma_\Omega^{-1}(\mathrm{d}f)) = \rho(\alpha)$$

is a complete vector field, so is X^α, by [1, Proposition 13.34]. ■

In the next section, we will try to realize several Poisson manifolds as the base space of some symplectic groupoid. To do so, we will now prove that if a groupoid structure on a symplectic realization exists, it must be unique. The useful and important thing about the proof of this fact, is that it gives us a recipe on how to construct the groupoid structure, if we are given a symplectic realization. The existence part takes a lot more work, so we omit it, and direct the reader to [1, Chapter 14].

Theorem 4.11. Let $\mu\colon (S, \Omega) \to (M, \pi)$ be a symplectic realization of a Poisson manifold M, with connected μ-fibres, and let $u\colon M \to S$ be a Lagrangian section of μ, i.e. $u(M)$ is a Lagrangian embedding of M into S, and $\mu \circ u = \mathrm{id}_M$. There is at most one groupoid structure on S, such that $(S, \Omega) \rightrightarrows M$ is a symplectic groupoid with target map μ and the unit map u. Moreover, such a groupoid structure exists if and only if the following holds:

(i) The symplectic realization is complete.

(ii) Each leaf of the foliation of the orthogonal distribution $(\ker \mathrm{d}\mu)^\Omega$ intersects $u(M)$ at precisely one point.

Proof. First off, we mention that the condition (i) is clearly necessary for the existence of the symplectic groupoid structure on S, by previous theorem.

Now suppose such a symplectic groupoid structure on S exists, and note that the target map is necessarily $t = \mu$ by assumption. Furthermore, t-fibres are connected, and s-fibres must also be connected since $s^{-1}(x) = \mathrm{inv}^{-1}(t^{-1}) = \mathrm{inv}(t^{-1}(x))$ is the image of a connected space along a continuous map. Since $\ker \mathrm{d}s = (\ker \mathrm{d}t)^\Omega$, s-fibres must be the leaves of foliation of the orbit distribution $(\ker \mathrm{d}t)^\Omega$, and this determines s entirely. The section $u\colon M \to S$ of t is also a section of s, since

$$x = t(u(x)) = s(\mathrm{inv}(u(x))) = s(u(x)),$$

and so each fibre $s^{-1}(x)$ of s intersects $u(M)$ at precisely one point, proving that condition (ii) is also necessary for existence of the groupoid structure.

We already have uniqueness of $t = \mu, s, u$. To show that multiplication is uniquely defined, note that the other structure maps determine the Lie algebroid

$$A := u^*(\ker \mathrm{d}\mu),$$

and also the Lie algebroid isomorphism

$$\sigma_\Omega\colon A \to T^*M, \quad \alpha \mapsto -u^*(\Omega^\flat(\alpha)).$$

The equation $\Omega^\flat(X^\alpha) = -s^*(\sigma_\Omega(\alpha))$ now determines left-invariant vector fields on S, and we can use this to write down the multiplication map, using their flows:[‖]

$$\phi_\lambda^{X^\alpha}(g) = L_g(\phi_\lambda^{X^\alpha}(1_{s(g)})),$$

that is, for any $g \in G$, there holds

$$m\big(g, \phi_\lambda^{X^\alpha}(1_{s(g)})\big) = \phi_\lambda^{X^\alpha}(g).$$

Now note that $\phi_\lambda^{X^\alpha}(1_{s(g)})$ is defined for all λ in some open inteval $J_{s(g)}^{X^\alpha}$ around zero, and hence the above formula determines multiplication on

$$S * U = \{(g, h) \in G \times U \mid s(g) = t(h)\},$$

where U is an open neighborhood of $u(M)$,

$$U = \{\phi_\lambda^{X^\alpha}(1_x) \mid \alpha \in \Gamma^\infty(A), \lambda \in J_x^{X^\alpha}, x \in M\}.$$

Since t-fibres are connected by assumption, the multiplication on the whole groupoid is determined by multiplication on U, as the next lemma proves. ■

Lemma 4.12. Let $G \rightrightarrows M$ be a Lie groupoid with connected t-fibres, and U an open neighborhood of $u(M)$. Any element of G can be written as a finite product of elements in U.

Proof. Denote $V_x^1 = G^x \cap U$, and for any $n \geq 2$,

$$V_x^n = \bigcup_{g \in G^x} L_g\big(V_{s(g)}^{n-1}\big).$$

An induction argument shows that the set V_x^n is open in G^x for any $n \in \mathbb{N}$, since $L_g \colon G^{s(g)} \to G^{t(g)}$ is a diffeomorphism for any $g \in G$. Clearly, $V_x := \cup_{n \in \mathbb{N}} V_x^n$ is the set of all elements in G^x that can be written as a product of finitely many elements from U, and this set is open in G^x.

To see that its complement is open in G^x, note that if $g \notin V_x$, then $gU^{-1} := \{gu^{-1} \mid s(u) = s(g), u \in U\}$ is a neighborhood of g in G^x, which does not intersect V_x, for if it did, there would exist a $u \in U$ such that $gu^{-1} \in V_x^n$ for some $n \in \mathbb{N}$, and then we would have $g = u_1 \dots u_n u$ for some $u_i \in U$, contradicting $g \notin V_x$.

So V_x is open and closed in G^x, hence by assumption of t-connectedness, $V_x = G^x$. ■

Remark 4.13. To conclude this section, let us note that the deepest results in the theory of symplectic groupoids in Poisson geometry are those which concern *integrability*. That is, given a Poisson manifold (M, π), when is it possible to construct the symplectic groupoid that is its symplectic realization? If the Lie algebroid $(T^*M, [\cdot,\cdot]_\pi, \pi^\sharp)$ is integrable,[**] then by Lie's first theorem, we can find a Lie groupoid $\Sigma \rightrightarrows M$ with simply connected t-fibres, such that there is an isomorphism $A(\Sigma) \xrightarrow{\sigma} T^*M$ of Lie algebroids. The result by Mackenzie and Xu [1, Theorem 14.29] shows that in this case, there is a unique multiplicative symplectic form $\Omega \in \Omega^2(\Sigma)$, with the property that the isomorphism σ_Ω as in Theorem 4.9, equals σ.

[‖] To see that the equation for the flow of X^α holds, note that it is not hard to show that if $\gamma_{1_x}^{X^\alpha}$ denotes the integral path of X^α starting at 1_x, then $L_g \circ \gamma_{1_x}^{X^\alpha}$ is the integral path of X^α starting at g (provided $s(g) = x$), and thus uniqueness of integral paths guarantees the wanted equality.

[**] A deep result by Crainic and Fernandes is that this is the case precisely when the monodromy groups are uniformly discrete. For details, see [2].

4.5 Examples of symplectic groupoid realizations of Poisson manifolds

In this last section, we will first provide symplectic groupoid realizations of the standard nondegenerate Poisson structure on $\mathbb{R}^{2n}$, and the zero Poisson structure. After that, we will construct two nontrivial examples, using the recipe provided by the proof of Theorem 4.11.

Example 4.14 (Symplectic manifolds)**.** Recall from Example 4.5 (i) that, given a symplectic manifold (M, ω), the pair groupoid $M \times M \rightrightarrows M$ with the symplectic form $\Omega = \mathrm{pr}_1^* \omega - \mathrm{pr}_2^* \omega$ is a symplectic groupoid.

In this case, the target map $t = \mathrm{pr}_1$ is manifestly Poisson. The Lie algebroid $A(M \times M)$ has been shown in Example 3.11 to be isomorphic to TM, and the isomorphism σ_Ω from Theorem 4.9 now equals $\sigma_\Omega = \omega^\flat$.

A concrete example is $\mathbb{R}^{2n}$ with the standard symplectic coordinates $(q^i, p_i)_{i=1}^n$, and the canonical Poisson structure is given by $\{p_i, q^i\} = 1$, $\{q^i, q^j\} = \{p_i, p_j\} = 0$. The symplectic groupoid realization of $\mathbb{R}^{2n}$ is obtained by considering $\mathbb{R}^{4n}$, with coordinates $(q^i, p_i)_{i=1}^{2n}$. The symplectic groupoid is given by:

$$
\begin{array}{c|l}
\mathbb{R}^{4n} & \Omega = \displaystyle\sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p_i - \sum_{i=n+1}^{2n} \mathrm{d}q^i \wedge \mathrm{d}p_i \\
{\scriptstyle s}\downdownarrows{\scriptstyle t} & t(q^i, p_i)_{i=1}^{2n} = (q^i, p_i)_{i=1}^{n} \\
 & s(q^i, p_i)_{i=1}^{2n} = (q^i, p_i)_{i=n+1}^{2n} \\
\mathbb{R}^{2n} & (q^i, p_i)_{i=1}^{2n} \cdot (q^i, p_i)_{i=n+1}^{3n} = ((q^i, p_i)_{i=1}^{n}, (q^i, p_i)_{i=2n+1}^{3n})
\end{array}
$$

Note that the partial multiplication is just the pair groupoid one. ♦

Example 4.15 (Zero Poisson structures)**.** Recall from Example 4.5 (ii) that the cotangent bundle T^*M of a given manifold M, endowed with the canonical symplectic form ω, is a symplectic groupoid $(T^*M, \omega) \rightrightarrows M$. In this case, the source and target maps coincide, so we get $\ker \mathrm{d}t = (\ker \mathrm{d}t)^\omega$, and the induced Poisson structure on M is the zero structure. The Lie algebroid of $T^*M \rightrightarrows M$ can be easily identified with T^*M, and the isomorphism $\sigma_\omega \colon T^*M \to T^*M$ from Theorem 4.9 is now just the identity map.

For a more specific example, on $M = \mathbb{R}^n$ with coordinates $(q^i)_{i=1}^n$, we find that the symplectic groupoid is given by:

$$
\begin{array}{c|l}
\mathbb{R}^{2n} & \Omega = \displaystyle\sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p_i \\
{\scriptstyle s}\downdownarrows{\scriptstyle t} & t(q^i, p_i)_{i=1}^{n} = s(q^i, p_i)_{i=1}^{n} = (q^i)_{i=1}^{n} \\
\mathbb{R}^{n} & (q^i, \tilde{p}_i)_{i=1}^{n} \cdot (q^i, p_i)_{i=1}^{n} = ((q^i)_{i=1}^{n}, (p_i + \tilde{p}_i)_{i=1}^{n})
\end{array}
$$

Note that partial multiplication is just addition in the fibres. ♦

As promised, we now use the proof of Theorem 4.11 to construct two nontrivial examples.

Example 4.16. Consider $\mathbb{R}^2$ with the Poisson structure given by

$$\{x, y\} = x.$$

First off, to obtain its symplectic realization (S, ω), note that $\dim S \geq 4$ since π_x vanishes at $x = 0$, so we impose the candidate manifold as $S = \mathbb{R}^4$. We follow the recipe in Remark 2.4, so we let $\mu(x, y, u, v) = (x, y)$, and impose the Poisson structure as

$$\{x, y\} = x, \quad \{y, v\} = 1, \quad \{x, u\} = \phi, \quad \{x, v\} = \{u, v\} = \{y, u\} = 0,$$

where $\phi\colon \mathbb{R}^4 \to \mathbb{R}$ is a function, to be determined; this imposition of structure functions is made by taking into account the requirement that π_ω has to be nondegenerate. To determine the function ϕ, we must take into account the Jacobi identity, and a straightforward computation gives

$$\phi = \mathrm{e}^{-v},$$

so we get

$$\pi_\omega = x\partial_x \wedge \partial_y + \partial_y \wedge \partial_v + \mathrm{e}^{-v}\partial_x \wedge \partial_u,$$

or equivalently,

$$\omega = \mathrm{e}^v(\mathrm{d}u \wedge \mathrm{d}x + x\,\mathrm{d}u \wedge \mathrm{d}v) + \mathrm{d}v \wedge \mathrm{d}y.$$

This can be expressed as $\omega = -\,\mathrm{d}(x\vartheta_1 + y\vartheta_2)$, where $\vartheta_1 = \mathrm{e}^v\,\mathrm{d}u$ and $\vartheta_2 = \mathrm{d}v$ form a coframe on $\mathbb{R}^2$, which satisfies $\mathrm{d}\vartheta_1 = -\vartheta_1 \wedge \vartheta_2, \mathrm{d}\vartheta_2 = 0$. It is clear that ω is a symplectic form on $\mathbb{R}^4$, and that the projection $\mu\colon (S,\omega) \to (M,\pi)$ onto the first two coordinates is a Poisson map, which is a surjective submersion.

To yield a groupoid structure on S, we follow the recipe in the proof of Theorem 4.11. We prescribe the Lagrangian section $u\colon \mathbb{R}^2 \to \mathbb{R}^4$ as the obvious inclusion $u(x,y) = (x,y,0,0)$, and consider the infinitesimal action $\mathfrak{a}\colon \Omega^1(\mathbb{R}^2) \to \mathfrak{X}^1(\mathbb{R}^4)$, which is by definition given (on the coframe $\mathrm{d}x, \mathrm{d}y$ of M) by

$$\mathfrak{a}(\mathrm{d}x) = \pi_\omega^\sharp(\mathrm{d}x) = x\partial_y + \mathrm{e}^{-v}\partial_u, \quad \mathfrak{a}(\mathrm{d}y) = \pi_\omega^\sharp(\mathrm{d}y) = -x\partial_x + \partial_v,$$

where we have used the definition of π_ω above. The flows of these vector fields read

$$\begin{aligned}\phi_\lambda^{\mathfrak{a}(\mathrm{d}x)}(x,y,u,v) &= (x, y + x^\lambda, u + \mathrm{e}^{-v}\lambda, v),\\ \phi_\lambda^{\mathfrak{a}(\mathrm{d}y)}(x,y,u,v) &= (\mathrm{e}^{-\lambda}x, y, u, \lambda + v),\end{aligned}$$

and now recall that $\mathrm{im}(\mathfrak{a}_p) = (\ker \mathrm{d}\mu_p)^\omega$, i.e. the image of $\mathfrak{a}$ spans the orbit distribution $(\ker \mathrm{d}\mu)^\omega$, and the s-fibres should be its integral manifolds. To compute s, hence note that any function $f\colon \mathbb{R}^4 \to \mathbb{R}$ which is constant on the image of $\mathfrak{a}$, must be of the form

$$f(x,y,u,v) = f(\mathrm{e}^{-\lambda}x, y, u, \lambda + v) = f(x, y + x\mu, u + \mathrm{e}^{-v}\mu, v)$$

for any $\lambda, \mu \in \mathbb{R}$. To get the point which intersects the image of the unit section $u\colon \mathbb{R}^2 \to \mathbb{R}^4$, we first set $\lambda = -v$, so that

$$f(x,y,u,v) = f(\mathrm{e}^v x, y, u, 0) = f(\mathrm{e}^v x, y + \mathrm{e}^v x\mu, u + \mu, 0),$$

and secondly, we set $\mu = -u$, so the source map should read

$$(x,y,u,v) \mapsto (x\mathrm{e}^{-v}, y - \mathrm{e}^v x). \tag{12}$$

Since μ is the simpler map, we rather set the source as $s = \mu$ and the target t as (12), but now we have to also change the sign of the symplectic form, so we set $\Omega = -\omega$. At this point, we also set $\Sigma = \mathbb{R}^4$. Finally, we have to find the partial multiplication on Σ which will make it into a symplectic groupoid. To that end, note that

$$s(\tilde{x},\tilde{y},\tilde{u},\tilde{v}) = t(x,y,u,v) \iff \tilde{x} = x\mathrm{e}^v \text{ and } \tilde{y} = y - xu\mathrm{e}^v,$$

and compute $t(\tilde{x},\tilde{y},\tilde{u},\tilde{v})$:

$$(\tilde{x}\mathrm{e}^{\tilde{v}}, \tilde{y} - \tilde{x}\tilde{u}\mathrm{e}^{\tilde{v}}) = t(\tilde{x},\tilde{y},\tilde{u},\tilde{v}) = t(x\mathrm{e}^v, y - xu\mathrm{e}^v, \tilde{u}, \tilde{v}) = (x\mathrm{e}^{v+\tilde{v}}, y - x(u\mathrm{e}^{-\tilde{v}} + \tilde{u})\mathrm{e}^{v+\tilde{v}}).$$

We can depict this situation diagrammatically as

$$\begin{tikzcd}
(x\mathrm{e}^{v+\tilde{v}}, y - x(u\mathrm{e}^{-\tilde{v}} + \tilde{u})\mathrm{e}^{v+\tilde{v}}) & (\tilde{x}, \tilde{y}) \arrow[l, "{(\tilde{x},\tilde{y},\tilde{u},\tilde{v})}"'] & (x, y) \arrow[l, "{(x,y,u,v)}"'] \arrow[ll, dashed, bend right, "{(x,y,U,V)}"']
\end{tikzcd}$$

This suggests that we take $V = v + \tilde{v}$ and $U = u\mathrm{e}^{-\tilde{v}} + \tilde{u}$. Summing up, the candidate for our symplectic groupoid (Σ, Ω) that realizes (M, π) is:

$$\begin{array}{c|l}
 & \Omega = \mathrm{e}^{v}(\mathrm{d}x \wedge \mathrm{d}u + x\,\mathrm{d}v \wedge \mathrm{d}u) + \mathrm{d}y \wedge \mathrm{d}v \\
\mathbb{R}^4 & s(x, y, u, v) = (x, y) \\
s \downdownarrows t & t(x, y, u, v) = (x\mathrm{e}^{-v}, y - \mathrm{e}^{v}x) \\
\mathbb{R}^2 & (\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}) \cdot (x, y, u, v) = (x, y, u\mathrm{e}^{-\tilde{v}} + \tilde{u}, v + \tilde{v})
\end{array}$$

It is not hard to show that the form Ω is indeed multiplicative, so that this is indeed our wanted symplectic groupoid realization.

Note also that this is an action groupoid of the Lie group $G = \mathbb{R}^2$ with the group multiplication given by

$$(\tilde{u}, \tilde{v})(u, v) = (u\mathrm{e}^{-\tilde{v}} + \tilde{u}, v + \tilde{v})$$

which acts on $\mathbb{R}^2$ from the left by $(u, v) \cdot (x, y) = (x\mathrm{e}^{v}, y - xu\mathrm{e}^{v})$. It can be checked that this action groupoid is precisely the symplectic coadjoint action groupoid from Example 4.5 (iii). ⧫

We list the following example without further justification; the groupoid structure is obtained in a similar fashion to the one above.

Example 4.17. The symplectic groupoid realization of the LV-type Poisson structure on $\mathbb{R}^2$,

$$\{x, y\} = axy, \quad (a \in \mathbb{R})$$

can be obtained similarly as in previous example. It turns out that it is the following one.

$$\begin{array}{c|l}
 & \Omega = \mathrm{d}x \wedge \mathrm{d}u + \mathrm{d}y \wedge \mathrm{d}v - a\,\mathrm{d}(xu) \wedge \mathrm{d}(yv) \\
\mathbb{R}^4 & s(x, y, u, v) = (x, y) \\
s \downdownarrows t & t(x, y, u, v) = (x\mathrm{e}^{ayv}, y\mathrm{e}^{-axu}) \\
\mathbb{R}^2 & (\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}) \cdot (x, y, u, v) = (x, y, u + \mathrm{e}^{ayv}\tilde{u}, v + \mathrm{e}^{-axu}\tilde{v})
\end{array}$$

For $a \neq 0$, this turns out to no longer be an action groupoid. ⧫

References

[1] Marius Crainic, Rui Loja Fernandes, and Ioan Mărcuţ. *Lectures on Poisson geometry.* AMS, 2021.

[2] Marius Crainic and Rui Loja Fernandes. "Lectures on integrability of Lie brackets". In: *Geom. Topol. Monogr* (2011), pp. 1–107.