Sheaves on topological spaces and toposes

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Abstract

Sheaves arise in areas of mathematics where objects being dealt with are of a local nature, in the sense that one can glue locally defined objects to yield global ones. In the present exposition, we show that this idea is elegantly described within the category theoretic framework, and demonstrate that the category of sheaves on a topological space is an elementary topos. We conclude our discussion with a view towards topos theory.

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1 Sheaves on topological spaces

1.1 Elementary definitions and examples

Throughout this document, X will denote a topological space. To motivate the definition of a sheaf, consider the algebra $C(U)$ of continuous real-valued functions on any open subset $U \subset X$. If $V \subset U$ is another open set in X, the restriction map $\rho_{U,V} := -|_V : C(U) \to C(V)$ satisfies the simple property that

$$(f|_V)|_W = f|_W \quad \text{for all } f \in C(U),$$

whenever W is yet another open subset with $W \subset V \subset U$. In other words, $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$, which may be rephrased categorically, as follows. Define the *category* $\mathscr{O}(X)$ *of open sets* on X by $\mathrm{Obj}(\mathscr{O}(X)) = \Omega(X)$, i.e. the set of its objects is the topology $\Omega(X)$ on X, and the morphisms are their inclusions, that is

$$\mathrm{Hom}(V,U) = \begin{cases} \{V \hookrightarrow U\} & \text{if } V \subset U, \\ \emptyset & \text{otherwise} \end{cases}$$

In this way, $C \colon \mathscr{O}(X) \to \mathbf{Set}$ becomes a contravariant functor, defined as

$$U \mapsto C(U), \quad (V \hookrightarrow U) \mapsto \rho_{U,V} \colon C(U) \to C(V).$$

Furthermore, any $f \in C(U)$ may be glued together from local pieces – more specifically, if $(U_i)_{i \in I}$ is any open cover of U in the sense that $U = \cup_i U_i$, and we are given functions $f_i \in C(U_i)$, then there is *at most one* $f \in C(U)$ with $f|_{U_i} = f_i$ for all $i \in I$, which *exists* if and only if $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, for any two $i, j \in I$.

Definition 1.1. A *presheaf* on X is a (covariant) functor $F \colon \mathscr{O}(X)^{\mathrm{op}} \to \mathbf{Set}$. The map $\rho_{U,V} := F(V \hookrightarrow U)$ is called the *restriction map* from U to V, and we write $f|_V := \rho_{U,V}(f)$ for any $f \in F(U)$.

Definition 1.2. A *sheaf* on X is a presheaf $F \colon \mathscr{O}(X)^{\mathrm{op}} \to \mathbf{Set}$ that satisfies the following two *collation conditions*:

(i) For any open subset $U \subset X$, open cover $(U_i)_{i \in I}$ of U, and $f, g \in F(U)$, if there holds $f|_{U_i} = g|_{U_i}$ for all $i \in I$, then $f = g$.

(ii) For any open subset $U \subset X$, open cover $(U_i)_{i \in I}$ of U, and elements $f_i \in F(U_i)$ with the property that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for any two $i, j \in I$.

Remark 1.3. The use of letter F is customary since the french translation of the word sheaf is *faisceau*. The above collation conditions are easily expressed in categorical framework in the following way. Given an open cover $(U_i)_{i \in I}$ of U, we have the canonical arrows in $\mathbf{Set}$:

$$e \colon F(U) \to \prod_i F(U_i), \qquad e(f) = (f|_{U_i})_i,$$

$$p, q \colon \prod_i F(U_i) \to \prod_{i.j} F(U_i \cap U_j), \quad p(f_i)_i = (f_i|_{U_i \cap U_j})_{i,j}, \quad q(f_i)_i = (f_j|_{U_i \cap U_j})_{i,j}.$$

The collation conditions are then expressed as: e is the equalizer of p and q, i.e. the diagram

$$F(U) \xrightarrow{\ e\ } \textstyle\prod_i F(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} F(U_i \cap U_j)$$

is equalizer in $\mathbf{Set}$.

Example 1.4.

(i) The functor $\mathbf{y}(X) = \mathrm{Hom}_{\mathscr{O}(X)}(-, X)$ is obviously a presheaf. To prove it is a sheaf, take any open subset $U \subset X$ and its open cover $(U_i)_{i \in I}$. Observe that elements of $\mathbf{y}(X)(U_i)$ are just open subsets of U_i and the restriction maps are intersections, so if we are given $V_i \subset U_i$ with the property that $V_i \cap U_j = V_j \cap U_i$ for all $i, j \in I$, then there is a unique open subset $V \subset X$ such that $V \cap U_i = V_i$, namely $V = \cup_i V_i$. This proves the second collation condition (the first one is trivial). This is sometimes called the *constant sheaf* on X, and we write $\mathbf{y}(X)(U) = \{U \hookrightarrow X\} = \{*\}$. In particular, if $X = \emptyset$, $\mathbf{y}(\emptyset) = \{*\}$, and any sheaf on $\emptyset$ must be of such form since the product over an empty (index) set is a singleton, so collation condition reads $F(\emptyset) \to \{*\} \rightrightarrows \{*\}$.

(ii) From the motivating example, we see that C is a sheaf on any topological space X. Furthermore, if X is a smooth manifold, the functor $C^k \colon \mathscr{O}(X)^{\mathrm{op}} \to \mathbf{Set}$, $k \in \mathbb{N} \cup \{\infty\}$ which assigns to any open subset U the set of k-times continuously differentiable functions, is a sheaf since differentiability is local. This means that we get a nested sequence of subsheaves $C^\infty \subset C^k \subset \dots C^1 \subset C$ (the notion of a subsheaf is explained in section 1.2). Similarly, we have the sheaf H of holomorphic functions on a complex manifold, which is a subsheaf of C^∞.

(iii) The presheaf B of bounded functions on a topological space X is a presheaf, but not a sheaf since bounded functions may fail to collate to a bounded function. Hence condition (ii) above fails, however, the uniqueness in condition (i) still holds – in other words, the map e is a monomorphism but not the equalizer of p and q. Such a presheaf is said to be *separated.*

Definition 1.5. A *morphism of presheaves* F and G on X is a natural transformation $\eta \colon F \Rightarrow G$, i.e. for any two open subsets $V \subset U$ in X, there must hold $\eta_U(f)|_V = \eta_V(f|_V)$ for all $f \in F(U)$. We will denote the (functor) category of presheaves by $[\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}]$, and its subcategory of sheaves by $\mathbf{Sh}(X)$.

Remark 1.6. The subcategory $\mathbf{Sh}(X)$ in $[\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}]$ is obviously full, since we're only restricting to objects where collation conditions hold. Moreover, if $g \colon X \to Y$ is a continuous map of topological spaces, defining a functor $g_* \colon \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ on objects as $(g_*F)(V) = F(g^{-1}(V))$ and on morphisms as $(g_*\eta)_V = \eta_{g^{-1}(V)}$, for any open $V \subset Y$, yields a functor $\mathrm{Sh} \colon \mathbf{Top} \to \mathbf{Cat}$.

The object $\mathbf{y}(X)$ in $\mathbf{Sh}(X)$ is clearly terminal, since if F is a sheaf on X, there is precisely one map $\eta_U \colon F(U) \to \mathbf{y}(X)(U) = \{*\}$, and η is manifestly a natural transformation.

1.2 Subsheaves

Subobjects in the (functor) category $[\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}]$ of presheaves are subfunctors – recall that if $F \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ is a functor, then a functor $G \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ is called a *subfunctor* of F, if:

(i) $G(c) \subset F(c)$ for all $c \in \mathrm{Obj}(\mathbf{C})$,

(ii) $G(f) = F(f)|_{G(c)}$ for all $f \colon d \to c$ in $\mathbf{C}$.

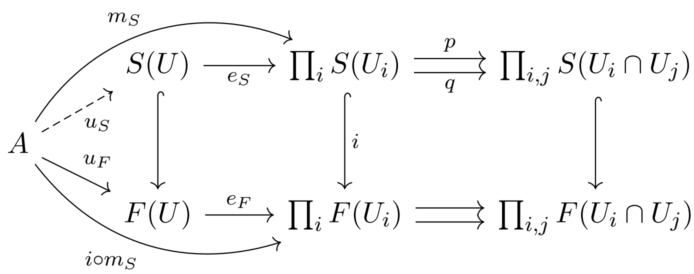
Definition 1.7. A *subsheaf* of F on X is a subfunctor of F which is itself a sheaf.

Proposition 1.8. A subfunctor S of a sheaf F on X is a subsheaf if and only if for any open subset $U \subset X$ and $f \in F(U)$ and any open cover $(U_i)_{i \in I}$ of U,

$$f \in S(U) \iff f|_{U_i} \in S(U_i) \text{ for all } i \in I.$$

Proof. If S is a sheaf, $f \in F(U)$ and $f|_{U_i} \in S(U_i)$ clearly imply $f \in S(U)$ by collation conditions.

Conversely, we draw the following diagram in **Set** (bottom row is an equalizer, vertical arrows are monomorphisms, m_S is an arbitrary map between sets with $p \circ m_S = q \circ m_S$, and u_F is the unique map such that $e_F \circ u_F = i \circ m_S$):



Here, we have two commutative squares on the right (in any of the two, either take the top or the bottom arrows). For any $a \in A$, write $m_S(a) = (m_S^i(a))_i$, where $m_S^i(a) \in S(U_i)$. We must have the equality $m_S^i(a) = u_F(a)|_{U_i}$ since the bottom row is an equalizer, hence $u_F(a)|_{U_i} \in S(U_i)$ and so by assumption the map u_F maps into $S(U)$, so we set $u_S = u_F$. Hence the top row is an equalizer as well. ∎

1.3 Sheaves of sections of bundles

Our next objective is to show that every sheaf can be seen as a sheaf of sections of a bundle. A *bundle* over X is a continuous map $p\colon Y \to X$. There is an abundance of examples of bundles, the most important of which are fibre bundles (vector bundles, principal G-bundles, associated bundles, and covering maps). Categorically, bundles over X are just objects of the slice category $\mathbf{Top}/X$; the morphisms between $p\colon Y \to X$ and $p'\colon Y' \to X$ in this category are continuous maps $g\colon Y \to Y'$ such that $p' \circ g = p$.

A *section* of a bundle $p\colon Y \to X$ is a map $s\colon U \to Y$ defined on an open subset $U \subset X$, such that $p \circ s$ is the inclusion $i_U\colon U \hookrightarrow X$. We write

$$\Gamma Y(U) = \{s\colon U \to Y \mid p \circ s = i_U\}$$

for the set of all sections of the bundle $p\colon Y \to X$. If $V \subset U$ is an open set of X, we get a restriction map $\Gamma Y(U) \to \Gamma Y(V)$, which means that $\Gamma Y\colon \mathscr{O}(X)^{\mathrm{op}} \to \mathbf{Set}$ is a presheaf. The same argument as in the case of the sheaf C of continuous functions on X shows that ΓY is in fact a sheaf over X, called the *sheaf of sections of* $p\colon Y \to X$. Furthermore, given a morphism of bundles $g\colon Y \to Y'$, we can define a morphism $\Gamma(g)\colon \Gamma Y \Rightarrow \Gamma Y'$ of sheaves by $\Gamma(g)_U(s) = g \circ s$, for any section $s\colon U \to Y$. In this way, $\Gamma\colon \mathbf{Top}/X \to \mathbf{Sh}(X)$ becomes a functor.

Examples of sheaves of sections include the sheaf of differential forms (more generally, tensor fields) on a smooth manifold, the sheaves of gauges and connections on principal bundles, and so on. Note also that the sheaf of continuous functions C is just the sheaf of sections of the bundle $\mathrm{pr}_X\colon X \times \mathbb{R} \to X$.

1.3.1 Bundle of germs of a presheaf

Let us demonstrate that every presheaf gives rise to a bundle. To do so, we introduce the notion of a germ.

Definition 1.9. Let $P\colon \mathscr{O}(X)^{\mathrm{op}} \to \mathbf{Set}$ be a presheaf, fix $x \in X$, and suppose U and V are two open neighborhoods of x. We say that $s \in P(U)$, $t \in P(V)$ have the *same germ*, if there is an open neighborhood $W \subset U \cap V$ of x such that $s|_W = t|_W$. This defines an equivalence relation on the set $\cup\{P(U) \mid \text{U is an open neighborhood of } x\}$, and any equivalence class $\mathrm{germ}_x(s)$ is called the *germ* of s at x.

The *stalk* of the presheaf P at $x \in X$ is the set

$$P_x = \{\mathrm{germ}_x(s) \mid s \text{ is a section, defined on a nbd. of } x\}$$

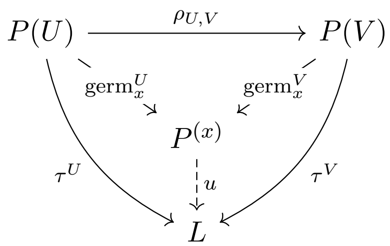
Proposition 1.10. The stalk P_x of any presheaf P on X is the colimit of the diagram $P^{(x)}$, defined as the restriction of P to the full subcategory

$$\mathscr{O}(X)^{\mathrm{op}}_x = \{U \in \Omega(X) \mid x \in U\}.$$

of $\mathscr{O}(X)^{\mathrm{op}}$. In other words,

$$P_x = \varinjlim_{U \ni x} P(U).$$

Proof. The object P_x in **Set**, together with maps $(\mathrm{germ}^U_x\colon P(U) \to P_x)_{U\ni x}$ form a cocone of the diagram $P^{(x)}$, since if $V \subset U$ is open in X, we have $\mathrm{germ}^V_x \circ \rho_{U,V} = \mathrm{germ}^U_x$. Furthermore, if $(\tau^U\colon P(U) \to L)_{U\ni x}$ is any other cocone of $P^{(x)}$, we have the following diagram



so we define $u(\mathrm{germ}_x s) = \tau^U(s)$ whenever $s \in P(U)$, hence our cocone is universal. ∎

We now define the *bundle of germs* of P as the set

$$\Lambda(P) = \coprod_{x\in X} P_x,$$

together with the canonical projection $p\colon \Lambda(P) \to M$, $p(\mathrm{germ}_x s) = x$. The task at hand now is to endow $\Lambda(P)$ with a topology. To do so, observe that any $s \in P(U)$ determines a map

$$\bar{s}\colon U \to \Lambda(P), \quad \bar{s}(x) = \mathrm{germ}_x s.$$

Proposition 1.11. A basis for some topology on $\Lambda(P)$ is given by the family

$$\mathcal{B} = \{\bar{s}(U) \mid U \subset X \text{ is open}, s \in P(U)\}.$$

With respect to this topology, the canonical projection p and the section $\bar{s}$, induced by an arbitrary $s \in P(U)$, are local homeomorphisms.

Proof. If $\bar{s}(U), \bar{t}(V) \in \mathcal{B}$ intersect, pick any $\mathrm{germ}_x s$ from the intersection, and now s and t must coincide on an open neighborhood $W \subset V \cap U$ of x (by definition of a germ), so $\bar{s}(W) \subset \bar{s}(U) \cap \bar{t}(V)$. This proves $\mathcal{B}$ is a basis for some topology on $\Lambda(P)$.

Given $s \in P(U)$, the map $\bar{s}\colon U \to \Lambda(P)$ is continuous since

$$\bar{s}^{-1}(\bar{t}(V)) = \{x \in U \cap V \mid \bar{s}(x) = \bar{t}(x)\} = \{x \in U \cap V \mid \mathrm{germ}_x s = \mathrm{germ}_x t\},$$

so by definition of a germ, for any $x \in U \cap V$, there exists an open neighborhood $W_x \subset U \cap V$ such that $W \subset \bar{s}^{-1}(\bar{t}(V))$. The map $\bar{s}$ is clearly open.

Finally, if $U \subset X$ is open, $p^{-1}(U) = \cup_{s \in P(U)} \bar{s}(U)$, so p is continuous. That p is open is clear from $p(\bar{s}(U)) = U$, which holds for any $s \in P(U)$, where $U \subset X$ is open. ■

Remark 1.12. A bundle $p\colon Y \to X$ is said to be *étale*, if p is a local homeomorphism. Last proposition shows that for any presheaf P, $\Lambda(P)$ is étale. The subcategory $\textbf{Étale}(X)$ of $\mathbf{Top}/X$ is clearly full, since we're only restricting the class of objects.

Typical examples of étale bundles are provided by covering spaces, however, not every étale bundle is a covering space, since it may fail to be locally trivial. For example, if $(U_i)_i$ is an open cover of X, the disjoint union $\coprod_i U_i$ with the obvious projection $\coprod_i U_i \to X$ is étale, but the fiber over a given point has cardinality the number of open sets U_i containing it, so $\coprod_i U_i$ does not have a typical fibre. Moreover, if $p\colon Y \to X$ is étale, the size of the open neighborhoods V_i of various points $y_i \in p^{-1}(x)$ that lie in the same fibre, as required in the definition of the étale bundle, may differ, and so the intersection $\cap_i p(V_i)$ of their projections may fail to be open.

To make $\Lambda\colon [\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}] \to \textbf{Étale}(X)$ into a functor, we need to describe it on morphisms of presheaves. If $\Theta\colon P \Rightarrow Q$ is a morphism of presheaves, it induces at each $x \in X$ a map

$$\Theta_x\colon P_x \to Q_x, \quad \Theta_x(\mathrm{germ}_x^P s) = \mathrm{germ}_x^Q(\Theta_U(s))$$

for any $s \in P(U)$, where U is any open neighborhood of x. Well-definedness of this map follows from naturality of Θ. Thus, for any $x \in X$, we get a functor

$$[\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Set},$$

given on objects as $P \mapsto P_x$ and on morphisms as $\Theta \mapsto \Theta_x$. Finally, the morphism Θ induces the map

$$\Lambda(\Theta)\colon \Lambda(P) \to \Lambda(Q),$$

defined as the disjoint union $\Lambda(\Theta) = \coprod_{x \in X} \Theta_x$, which is continuous since

$$\Lambda(\Theta)^{-1}(\bar{s}(U)) = \cup_{x \in U}\{\mathrm{germ}_x^P(t) \mid t \in P(U), \Theta_x(\mathrm{germ}_x^P(t)) = \mathrm{germ}_x^Q(s)\},$$

where $s \in Q(U)$ is arbitrary; this set is open in $\Lambda(P)$ since for any $\mathrm{germ}_x^P(t)$ from it, there is an open neighborhood $W \subset U$ of x such that $\Theta_U(t)|_W = s|_W$, hence $\bar{t}(W)$ is the open neighborhood of $\mathrm{germ}_x^P(t)$ contained in $\Lambda(\Theta)^{-1}(\bar{s}(U))$. This proves the following proposition.

Proposition 1.13. The map $\Lambda\colon [\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}] \to \textbf{Étale}(X)$, which takes any presheaf into its bundle of germs, is a functor.

Remark 1.14. The bundle $\Lambda(P)$ over X isn't necessarily Hausdorff, even if X is. For example, take the sheaf $P = C$ of continuous functions on $\mathbb{R}$, and consider the functions $f = 0$ and

$$h(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Then $\mathrm{germ}_0 f \neq \mathrm{germ}_0 h$, but $\mathrm{germ}_t f = \mathrm{germ}_t h$ for all $t < 0$, so any neighborhood of $\mathrm{germ}_0 f$ intersects every neighborhood of $\mathrm{germ}_0 h$ in $\mathrm{germ}_t f$ for some $t < 0$.

In contrast, considering the case of the sheaf H of holomorphic functions on $\mathbb{C}$, $\Lambda(H)$ is Hausdorff due to the identity principle for holomorphic functions.

1.3.2 Sheafification of a presheaf

The next important step is to consider the sheaf

$$\Gamma\Lambda(P)$$

of sections of the bundle $\Lambda(P) \to X$ of germs of a given presheaf P. On any open subset $U \subset X$, we define a morphism $\eta_P \colon P \Rightarrow \Gamma\Lambda(P)$ of presheaves as

$$(\eta_P)_U \colon P(U) \to \Gamma\Lambda P(U), \quad (\eta_P)_U(s) = \bar{s},$$

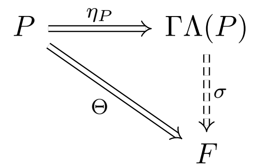
which is indeed a natural transformation since $(\eta_P)_U(s)|_V = \bar{s}|_V = \overline{s|_V} = (\eta_P)_V(s|_V)$ for any open subset $V \subset U$. In what follows, we show that $\Gamma\Lambda$ is an optimal solution to the problem of producing a sheaf out of a presheaf P, i.e. that $\Gamma\Lambda$ is part of an adjunction.

Lemma 1.15. If the presheaf F is a sheaf, then η_F is a natural isomorphism. In other words, any sheaf F is naturally isomorphic to its sheaf $\Gamma\Lambda(F)$ of sections of its bundle of germs.

Proof. Let $U \subset X$ be open. We need to show that $(\eta_F)_U$ is an isomorphism in **Set**. Let $h \colon U \to \Lambda(F)$ be a cross section of $\Lambda(F)$; we will show that there is exactly one $s \in P(U)$, such that $h = \bar{s}$. By definition of the bundle of germs of F, there exists for any $x \in U$ an open neighborhood $U_x \subset U$ and $s_x \in F(U_x)$, so that $h(x) = \mathrm{germ}_x(s_x)$. Since h is continuous and $\overline{s_x}(U_x) \subset \Lambda(F)$ is open, there is an open neighborhood $V_x \subset U$ of x, such that $h(V_x) \subset \overline{s_x}(U_x)$. The sets $(V_x)_{x\in U}$ clearly cover U, and the sections $\overline{s_x}, \overline{s_y}$ both agree with h on $V_x \cap V_y$ for any two points $x, y \in U$, thus there holds $\mathrm{germ}_z(s_x) = \mathrm{germ}_z(s_y)$ for any $z \in V_x \cap V_y$. The collation conditions of the sheaf F now ensure that there is a unique $s \in P(U)$, such that $s|_{V_x} = s_x$ for all $x \in U$. By construction, $h = \bar{s}$. ∎

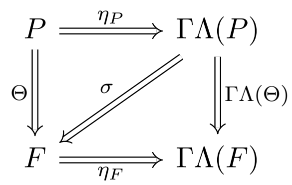
Let us now show that the inclusion functor $I \colon \mathbf{Sh}(X) \to [\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}]$ of sheaves into presheaves on X, is the right adjoint of $\Gamma\Lambda$, which will justify why the functor $\Gamma\Lambda$ is called the *sheafification functor*.

Theorem 1.16. For any presheaf P on X, the pair $\big(\Gamma\Lambda(P), \eta_P \colon P \Rightarrow I(\Gamma\Lambda(P))\big)$ is the universal morphism from P to the inclusion functor $I \colon \mathbf{Sh}(X) \to [\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}]$. That is, for any sheaf F on X and any morphism $\Theta \colon P \Rightarrow F$ of presheaves, there is a unique morphism $\sigma \colon \Gamma\Lambda(P) \Rightarrow F$ of sheaves, such that $\sigma \circ \eta_P = \Theta$:



Remark 1.17. Abuse of notation: η_P denotes the morphism $I \circ \eta_P$ of presheaves.

Proof. Since $\eta_F \colon F \Rightarrow \Gamma\Lambda(F)$ is an isomorphism, we may define the morphism σ of sheaves as $\sigma = \eta_F^{-1} \circ \Gamma\Lambda(\Theta)$, i.e. as the bottom triangle of the diagram:



That the top triangle also commutes is a straightforward computation:

$$\sigma_U((\eta_P)_U(s)) = \sigma_U(\bar{s}) = (\eta_F)_U^{-1}(\underbrace{\Gamma\Lambda(\Theta)(\bar{s})}_{\Lambda(\Theta)\circ\bar{s}}) = \Theta_U(s),$$

where $s \in P(U)$ was arbitrary.

To show uniqueness of σ, suppose $\tau\colon \Gamma\Lambda(P) \Rightarrow F$ is a morphism of sheaves with $\tau \circ \eta_P = \Theta = \sigma \circ \eta_P$. If $h\colon U \to \Lambda(P)$ is a section, then for any $x \in U$ there is a neighborhood $V_x \subset U$ and $s_x \in P(V_x)$, such that $h(x) = \mathrm{germ}_x(s_x)$. As in proof of Lemma 1.15, by continuity of h we can assume V_x is small enough for $h|_{V_x} = \overline{s_x} = (\eta_P)_{V_x}(s_x)$ to hold. By assumption and the naturality of σ and τ, we thus have $\sigma_U(h)|_{V_x} = \tau_U(h)|_{V_x}$. Since $U = \cup_{x\in U} V_x$ and F is a sheaf, we conclude $\sigma_U(h) = \tau_U(h)$ holds for any section $h\colon U \to \Lambda(P)$ and any open subset $U \subset X$, hence $\sigma = \tau$. ■

Corollary 1.18. For any space X, $\mathbf{Sh}(X)$ is a reflective subcategory of $[\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}]$. More precisely, we have the adjunction

$$\mathbf{Sh}(X) \underset{I}{\overset{\Gamma\Lambda}{\leftrightarrows}} [\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}]$$

of sheafification and inclusion functors ($\Gamma\Lambda$ is the left adjoint of I).

Proof. The only thing left to show is that $\eta\colon 1_{[\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}]} \Rightarrow I \circ (\Gamma\Lambda)$ is a natural transformation. One thus has to check $\eta_Q \circ \Theta = \Gamma\Lambda(\Theta) \circ \eta_P$, for any morphism $\Theta\colon P \Rightarrow Q$ of presheaves; this boils down to the definition $\Theta_x(\mathrm{germ}_x^P s) = \mathrm{germ}_x^Q(\Theta_U s)$ of $\Theta_x\colon P_x \to Q_x$, which holds for any section s, defined on an open neighborhood U of some $x \in X$. ■

1.3.3 Étalification of a bundle

For any bundle $p\colon Y \to X$ over X, recall from Proposition 1.11 that the bundle $\Lambda\Gamma(Y)$ is étale, and define the morphism of bundles $\varepsilon_Y\colon \Lambda\Gamma(Y) \to Y$ as

$$\varepsilon_Y(\mathrm{germ}_x s) = s(x) \quad \text{for any section } s\colon U \to Y,$$

which is a well defined map (by the definition of a germ), and it is continuous. Indeed, if $V \subset Y$ is open, then

$$\varepsilon_Y^{-1}(V) = \{\mathrm{germ}_{p(y)} s \mid y \in V, s \in \mathrm{sect}_Y(y)\}$$

where $\mathrm{sect}_Y(y)$ denotes the set of all sections of Y, defined on an open neighborhood of $p(y)$ in X and satisfying $s(p(y)) = y$. For $\mathrm{germ}_{p(y)} s \in \varepsilon_Y^{-1}(V)$, by continuity of s there is an open neighborhood U of $p(y)$ such that $s(U) \subset V$, hence $\varepsilon_Y(\bar{s}(U)) = s(U) \subset V$, so $\bar{s}(U)$ is an open neighborhood of $\mathrm{germ}_{p(y)} s$ in $\Lambda\Gamma(Y)$ that is contained in $\varepsilon_Y^{-1}(V)$.

Similarly to the result in Lemma 1.15, we now have the following result.

Lemma 1.19. If the bundle $p\colon Y \to X$ is étale, then ε_Y is a homeomorphism. In other words, any étale bundle is isomorphic to the bundle of the germs of its sections.

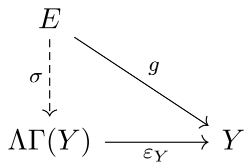
Proof. Let's construct an inverse $\rho_Y\colon Y \to \Lambda\Gamma(Y)$ of the map ε_Y. Fix $y \in Y$; since the bundle $p\colon Y \to X$ is étale, there exists an open neighborhood U of $p(y)$ in X and a section $s\colon U \to Y$ with $s(p(y)) = y$, i.e. $s \in \mathrm{sect}_Y(y)$. Moreover, if $t \in \mathrm{sect}_Y(y)$ is another such section of Y, defined on an open neighborhood V of $p(y)$, then t and s must coincide on an open neighborhood

$W \subset U \cap V$ of x, since the equalizer $\mathrm{Eq}_{s,t} = \{x \in U \cap V \mid s(x) = t(x)\} = p(s(U) \cap t(V))$ of the sections s, t is open in X.

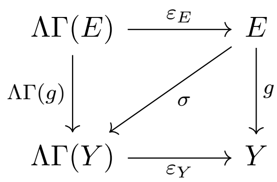
Defining $\rho_Y(y) = \bar{s}(p(y)) = \mathrm{germ}_{p(y)} s$, where $s \in \mathrm{sect}_Y(y)$ is arbitrary, hence yields a well-defined map, and this is clearly the inverse of ε_Y. Continuity of ε_Y follows from the fact that both $\tilde{p}\colon \Lambda\Gamma(Y) \to X$ and $p\colon Y \to X$ are local homeomorphisms, and there holds $p \circ \varepsilon_Y = \tilde{p}$, hence bijectivity and continuity of ε_Y imply it is a homeomorphism. ■

Dualizing sheafification, we can now show that the inclusion functor $J\colon \textbf{Étale}(X) \to \mathbf{Top}/X$ of étale bundles into bundles over X, is the left adjoint of $\Lambda\Gamma$, so that the functor $\Lambda\Gamma$ may be called the *étalification functor.*

Theorem 1.20. For any bundle $p\colon Y \to X$, the pair $(\Lambda\Gamma(Y), \varepsilon_Y\colon \Lambda\Gamma(Y) \to Y)$ is the universal morphism from the inclusion functor $J\colon \textbf{Étale}(X) \to \mathbf{Top}/X$ to Y. That is, for any étale bundle $q\colon E \to X$ and any morphism $g\colon E \to Y$ of bundles, there is a unique morphism $\sigma\colon E \to \Lambda\Gamma(Y)$ of étale bundles, such that $\varepsilon_Y \circ \sigma = g$:



Proof. We now consider the diagram below, and define a map $\sigma = \Lambda\Gamma(g) \circ \varepsilon_E^{-1}$, i.e. $\sigma(e) = \mathrm{germ}^Y_{q(e)}(g \circ s)$, where $s \in \mathrm{sect}_E(e)$ is arbitrary. This map is continuous, since ε_E is a homeomorphism.



The lower triangle commutes, since for any $e \in E$,

$$\varepsilon_Y(\sigma(e)) = \varepsilon_Y(\mathrm{germ}^Y_{q(e)}(g \circ s)) = (g \circ s)(q(e)) = g(e),$$

where $s \in \mathrm{sect}_E(e)$, as above.

To show uniqueness, suppose $\tau\colon E \to \Lambda\Gamma(Y)$ is a morphism of bundles with $\eta_Y \circ \tau = g = \eta_Y \circ \sigma$. Pick any $e \in E$ and denote $x = q(e)$. Since E is étale, we can pick $s \in \mathrm{sect}_E(e)$, and since τ is a bundle morphism, we must have

$$\tau(e) = \tau(s(x)) = \mathrm{germ}^Y_x(s^\tau)$$

for some section $s^\tau\colon U \to Y$, defined on an open neighborhood U of x in X. By assumption, we have $s^\tau(x) = \varepsilon_Y(\mathrm{germ}^Y_x(s^\tau)) = \varepsilon_Y(\mathrm{germ}^Y_x(g \circ s)) = (g \circ s)(x)$, and since the bundle $\Lambda\Gamma(Y)$ is also étale, this implies (as in proof of previous lemma) $s^\tau = g \circ s$ on a neighborhood on x, which is what we wanted to show. ■

Corollary 1.21. For any space X, $\textbf{Étale}(X)$ is a coreflective subcategory of $\mathbf{Top}/X$. More precisely, we have the adjunction

$$\mathbf{Top}/X \underset{\Lambda\Gamma}{\overset{J}{\leftrightarrows}} \textbf{Étale}(X)$$

of inclusion and étalification functors (J is left adjoint to $\Lambda\Gamma$).

Proof. The only thing left to prove is that $\varepsilon\colon J\circ(\Lambda\Gamma)\Rightarrow 1_{\mathbf{Top}/X}$ is a natural transformation. We thus have to check $g\circ\varepsilon_Y=\varepsilon_{Y'}\circ\Lambda\Gamma(g)$, for any morphism $g\colon Y\to Y'$ of bundles:

$$g(\varepsilon_Y(\mathrm{germ}_x^Y s))=g(s(x))=\mathrm{germ}_x^{Y'}(g\circ s)=\varepsilon_{Y'}(\Lambda\Gamma(g)(\mathrm{germ}_x^Y s)),$$

where s is a section of Y, defined on a neighborhood of $x\in X$. ∎

1.3.4 Equivalence of categories of sheaves and étale bundles

The jigsaw of the previous sections falls into place with the following result.

Theorem 1.22. For any space X, we have the adjunction

$$\mathbf{Top}/X \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} [\mathscr{O}(X)^{\mathrm{op}},\mathbf{Set}],$$

and the functors Λ and Γ restrict to an equivalence of categories

$$\mathbf{\acute{E}tale}(X) \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} \mathbf{Sh}(X).$$

Proof. We first show that $\eta\colon 1_{[\mathscr{O}(X)^{\mathrm{op}},\mathbf{Set}]}\Rightarrow\Gamma\Lambda$ and $\varepsilon\colon\Lambda\Gamma\Rightarrow 1_{\mathbf{Top}/X}$ are the unit and counit of the adjunction, respectively. Corollaries 1.18 and 1.21 show that they are natural transformations, so the only thing left to check is the triangle identities, i.e. that both compositions

$$\Gamma\xrightarrow{\eta\Gamma}\Gamma\Lambda\Gamma\xrightarrow{\Gamma\varepsilon}\Gamma,\qquad\qquad \Lambda\xrightarrow{\Lambda\eta}\Lambda\Gamma\Lambda\xrightarrow{\varepsilon\Lambda}\Lambda$$

are identities; that is, we should check $\Gamma(\varepsilon_Y)\circ\eta_{\Gamma Y}=1_{\Gamma Y}$ and $\varepsilon_{\Lambda P}\circ\Lambda(\eta_P)=1_{\Lambda P}$. We have:

$$\Gamma(\varepsilon_Y)(\eta_{\Gamma Y}(s))=\Gamma(\varepsilon_Y)(\bar{s})=\varepsilon_Y\circ\bar{s}=s,$$
$$\varepsilon_{\Lambda P}(\Lambda(\eta_P)(\mathrm{germ}_x f))=\varepsilon_{\Lambda P}((\eta_P)_x(\mathrm{germ}_x f))=\varepsilon_{\Lambda P}(\mathrm{germ}_x\bar{f})=\bar{f}(x)=\mathrm{germ}_x f,$$

where $s\colon U\to Y$ and $f\colon U\to\Lambda P$ are sections defined on an open subset $U\subset X$, and $x\in U$. Note that for clarity, we have omitted the subscripts U.

The second part of the theorem follows from the fact that restrictions of η and ε to $\mathbf{Sh}(X)$ and $\mathbf{\acute{E}tale}(X)$, respectively, are natural isomorphisms by Lemmas 1.15 and 1.19, hence the respective restrictions of the two functors Λ and Γ are quasi-inverse. ∎

Remark 1.23. The second part of last theorem means that each sheaf F on X can be considered as the étale bundle $\Lambda F\to X$; elements $s\in F(U)$ can be identified as sections $s\colon U\to\Lambda F$, the identification between the two given as $s\mapsto\bar{s}$; for an open subset $V\subset U$, the restriction map $FU\to FV$ can be seen as the actual restriction of the section. Similarly, every étale bundle Y over X can be seen as the sheaf ΓY on X.

1.4 Stalks of sheaves and the skyscraper sheaf

For completeness, we now expose some more properties of stalks of sheaves. We have already defined the stalk P_x of a presheaf P at $x\in X$. For any $x\in X$, we now define the *stalk functor* at x, denoted $\mathrm{Stalk}_x\colon[\mathscr{O}(X)^{\mathrm{op}},\mathbf{Set}]\to\mathbf{Set}$, given on objects as $F\mapsto F_x$ and on morphisms of sheaves as $\Theta\mapsto\Theta_x$, and it is not hard to show it is indeed a functor. In what follows, we will denote by Stalk_x the restriction of this functor to the subcategory $\mathbf{Sh}(X)$ of $[\mathscr{O}(X)^{\mathrm{op}},\mathbf{Set}]$.

Now let $A \in \mathrm{Obj}(\mathbf{Set})$ by any set and define the presheaf $\mathrm{Sky}_x(A) \colon \mathscr{O}(X)^{\mathrm{op}} \to \mathbf{Set}$ as

$$\mathrm{Sky}_x(A)(U) = \begin{cases} A & \text{if } x \in U, \\ \{*\} & \text{otherwise,} \end{cases}$$

where $\{*\}$ denotes some fixed singleton set. If $V \subset U$ is an open subset, we define

$$\mathrm{Sky}_x(A)(V \hookrightarrow U) = \begin{cases} \mathrm{id}_A & \text{if } x \in V, \\ \mu & \text{otherwise,} \end{cases}$$

where μ denotes the unique map into $\{*\}$.

Lemma 1.24. For any set A and any point $x \in X$, $\mathrm{Sky}_x(A)$ is a sheaf.

Proof. If $U \subset X$ is open and $(U_i)_i$ is its open cover, then let $f_i \in \mathrm{Sky}_x(A)(U_i)$, i.e.

$$f_i = \begin{cases} a_i \in A & \text{if } x \in U_i, \\ * & \text{otherwise.} \end{cases}$$

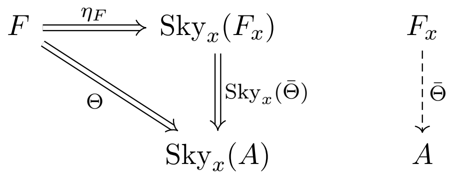
The assumption that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ forces $a_i = a_j$ whenever $x \in U_i \cap U_j$, so there is a unique $a \in A$ such that $a_i = a$ whenever $x \in U_i$. Hence elements f_i collate to a unique element $f \in \mathrm{Sky}_x(U)$ which is $f = a$ if $x \in U$, and $*$ otherwise. ■

One should think of the skyscraper sheaf as the sheaf, supported at a single point. The above construction now yields a functor $\mathrm{Sky}_x \colon \mathbf{Set} \to \mathbf{Sh}(X)$, called the *skyscraper functor* at $x \in X$, by acting on any arrow $g \colon A \to B$ in $\mathbf{Set}$ as $\mathrm{Sky}_x(g) \colon \mathrm{Sky}_x(A) \Rightarrow \mathrm{Sky}_x(B)$, which is pointwise determined by

$$\mathrm{Sky}_x(g)_U = \begin{cases} g & \text{if } x \in U, \\ \mu & \text{otherwise.} \end{cases}$$

Proposition 1.25. For any $x \in X$, the stalk functor $\mathrm{Stalk}_x \colon \mathbf{Sh}(X) \to \mathbf{Set}$ is left adjoint to the skyscraper functor Sky_x.

Proof. For any sheaf F, we will find a universal morphism η_F from F to Sky_x. That is, for any set A and a morphism $\Theta \colon F \Rightarrow \mathrm{Sky}_x(A)$ of sheaves, we have the following diagram.



We define η_F pointwise as the map $(\eta_F)_U \colon F(U) \to \mathrm{Sky}_x(F_x)(U)$,

$$(\eta_F)_U(f) = \begin{cases} \mathrm{germ}_x f & \text{if } x \in U, \\ * & \text{otherwise,} \end{cases}$$

and now determine the map $\bar{\Theta}$ by the above diagram, that is, $\mathrm{Sky}_x(\bar{\Theta})_U(\eta_F)_U(f) = \Theta_U(f)$. If $x \notin U$, $\bar{\Theta} = \mu$ is the only map that satisfies this condition. Otherwise, this condition reads $\bar{\Theta}(\mathrm{germ}_x f) = \Theta_U(f)$ for any $f \in F(U)$. This ensures uniqueness of $\bar{\Theta}$; we now of course use this to define $\bar{\Theta}$. Finally, it is not hard to check that $\eta \colon 1_{\mathbf{Sh}(X)} \Rightarrow \mathrm{Sky}_x \circ \mathrm{Stalk}_x$ is a natural transformation. ■

Remark 1.26. One consequence of the last proposition is that a morphism $\Theta\colon F \Rightarrow G$ of sheaves is a monomorphism (resp. epimorphism) if and only if for any $x \in X$, the map of stalks $\Theta_x\colon F_x \to G_x$ is injective (resp. surjective). Since we are not going to use this, we leave it as an exercise to the reader.

1.5 Properties of the category of sheaves

The objective of this section is to show that for any topological space X, the category $\mathbf{Sh}(X)$ is an elementary topos, i.e.:

(i) $\mathbf{Sh}(X)$ has all finite limits and colimits,

(ii) $\mathbf{Sh}(X)$ is cartesian closed,

(iii) $\mathbf{Sh}(X)$ has a subobject classifier.

By Theorem 1.22, the same then holds for **Étale**(X).

1.5.1 $\mathbf{Sh}(X)$ is complete and cocomplete

To show that $\mathbf{Sh}(X)$ is closed for small limits, we introduce the notion of a sieve, which will enable us to describe sheaves entirely in terms of objects of the category of presheaves. Intuitively, these are sets of arrows which exhibit similar behaviour as right ideals in ring theory.

Definition 1.27. A *sieve* on the object c of a category $\mathbf{C}$ is a set S of arrows to c, such that for any $f \in S$ and an arrow h in $\mathbf{C}$ with $\mathrm{cod}(h) = \mathrm{dom}(f)$, there holds $fh \in S$.

Proposition 1.28. For any object c in a small category $\mathbf{C}$, there is a bijective correspondence between sieves on c and subfunctors of $\mathbf{y}(c) = \mathrm{Hom}_{\mathbf{C}}(-, c)\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$.

Proof. Given a subfunctor Q of $\mathbf{y}(c)$, we have $Q(d) \subset \mathrm{Hom}_{\mathbf{C}}(d, c)$, so we define

$$S_Q = \bigcup_{d \in \mathrm{Obj}(\mathbf{C})} Q(d)$$

and this is a sieve, since if $f \in Q(d)$ and $h\colon d' \to d$ is any arrow in $\mathbf{C}$, then

$$Q(h) = \mathrm{Hom}(h, c)|_{Q(d)}\colon Q(d) \to Q(d'), \quad Q(h)(f) = fh \in Q(d') \subset S_Q$$

Conversely, if S is a sieve on c, define the functor Q_S on objects of $\mathbf{C}$ as

$$Q_S(d) = S \cap \mathrm{Hom}_{\mathbf{C}}(d, c) = \{f\colon d \to c \mid f \in S\}$$

and on arrows as $Q_S(h\colon d' \to d)\colon Q_S(d) \to Q_S(d')$, $Q_S(h)(f) = fh$, which is now clearly a subfunctor of $\mathbf{y}(c)$, and the maps $Q \mapsto S_Q$ and $S \mapsto Q_S$ are inverse. ■

Example 1.29. In the case $\mathbf{C} = \mathscr{O}(X)$, we have by the previous proposition that a sieve on an open set $U \subset X$ is a subfunctor S of $\mathbf{y}(U) := \mathrm{Hom}(-, U)$, i.e.

$$\mathbf{y}(U)(V) = \begin{cases} 1 & \text{if } V \subset U, \\ \emptyset & \text{otherwise,} \end{cases}$$

where 1 denotes the singleton $\{V \hookrightarrow U\}$. For example, if we set $S(V) = 1$ for some $V \subset U$, the condition (ii) of subfunctoriality of S forces $S(W) = 1$ for all open subsets $W \subset V$. Stated differently, a sieve S on U is a subset $S \subset \Omega(X)$ such that $W \subset V \in S$ implies $W \in S$, for any open subset $W \subset X$. We say that a sieve S on U is a *covering sieve* if $U = \cup_{V \in S} V$.

Theorem 1.30. A presheaf P on X is a sheaf if and only if for any open subset $U \subset X$ and every covering sieve S on U, the canonical inclusion $i_S \colon S \Rightarrow \mathbf{y}(U)$ of functors induces an isomorphism

$$\mathrm{Nat}(\mathbf{y}(U), P) \cong \mathrm{Nat}(S, P).$$

Remark 1.31. The induced map $i_S^* \colon \mathrm{Nat}(\mathbf{y}(U), P) \cong \mathrm{Nat}(S, P)$ is just $\eta \mapsto \eta \circ i_S$. Here, i_S is given as $(i_S)_V \colon S(V) \hookrightarrow \mathbf{y}(U)(V)$. We are abusing notation and writing $S = Q_S$, with Q_S as in previous proposition.

Proof. We will prove both directions at the same time, by inspecting the equalizer diagram in **Set** of the presheaf P:

$$E \xrightarrow{\;d\;} \textstyle\prod_i P(U_i) \overset{p}{\underset{q}{\rightrightarrows}} \textstyle\prod_{i,j} P(U_i \cap U_j)$$

where $(U_i)_i$ is any open cover of U. The upshot of the argument that follows is that E is isomorphic to $\mathrm{Nat}(S, P)$ for some covering sieve S on U, and the rest follows from the Yoneda lemma.

Since this is an equalizer in **Set**, we have

$$E = \left\{ (f_i)_i \in \prod_i P(U_i) \;\middle|\; f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \text{ for all } i, j \right\},$$

and d is the inclusion. We now replace the cover $(U_i)_i$ in this set with the covering sieve

$$S = \{V \in \Omega(X) \mid V \subset U_i \text{ for some } i\}$$

on U, in the following way. Define $f_V = f_i|_V$ and notice that since the elements f_i coincide on intersections $U_i \cap U_j$, the definition of f_V is independent of the index i. The equalizer E is thus in bijection with

$$E = \left\{ (f_V)_{V \in S} \in \prod_{V \in S} P(V) \;\middle|\; f_V|_{V'} = f_{V'} \text{ for all } V' \subset V \right\},$$

and the map d now maps $(f_V)_{V \in S} \mapsto (f_{U_i})_i \in \prod_i P(U_i)$ (note that the sets U_i are in S, and $f_{U_i} = f_i$). By previous proposition, we may regard S as a functor $\mathscr{O}(X)^{\mathrm{op}} \to \mathbf{Set}$,

$$S(V) = \begin{cases} 1 & \text{if } V \in S, \\ \emptyset & \text{if } V \notin S, \end{cases}$$

where $1 = \{V \hookrightarrow U\}$. In this way, for any $V \in S$, $f_V \in P(V)$ can be (trivially) interpreted as a map $S(V) \to P(V)$ which maps $1 \mapsto f_V$. This implies that the set E is in bijection with $\mathrm{Nat}(S, P)$, since any $(f_V)_{V \in S} \in E$ determines a unique morphism of presheaves $\Theta \colon S \Rightarrow P$ given by $\Theta_V(1) = f_V$, and the map $d \colon \mathrm{Nat}(S, P) \to \prod_i P(U_i)$ becomes $d(\Theta) = (\Theta_{U_i}(1))_i$. We have just shown that the equalizer of canonical maps p, q of a presheaf is $\mathrm{Nat}(S, P)$. Now augment the equalizer diagram above, by using i_S:

$$\begin{array}{ccccc}
\mathrm{Nat}(S, P) & \xrightarrow{\;d\;} & \prod_i P(U_i) & \overset{p}{\underset{q}{\rightrightarrows}} & \prod_{i,j} P(U_i \cap U_j) \\
\uparrow{\scriptstyle i_S^*} & & \uparrow{\scriptstyle e} & & \\
\mathrm{Nat}(\mathbf{y}(U), P) & \underset{y}{\xrightarrow{\;\cong\;}} & P(U) & &
\end{array}$$

Here, y denotes the Yoneda isomorphism, $y(\eta) = \eta_U(1_U)$. Notice that the square commutes:

$$\begin{aligned} e(y(\eta)) &= e(\eta_U(1_U)) = (\eta_U(1_U)|_{U_i})_i = (\eta_{U_i}(1_{U_i}))_i \\ d(i_S^*(\eta)) &= d(\eta \circ i_S) = ((\eta_{U_i} \circ (i_S)_{U_i})(1_{U_i}))_i = (\eta_{U_i}(1_{U_i}))_i, \end{aligned}$$

which implies that e is the equalizer of p, q if and only if $(i_S)^*$ is an isomorphism. ■

Proposition 1.32. For any space X, $\mathbf{Sh}(X)$ is complete.

Proof. It is enough to show that $\mathbf{Sh}(X)$ is closed for equalizers and small products (see [1, Chapter V, Section 2, Corollary 2]).

We first show that $\mathbf{Sh}(X)$ is closed for equalizers. Suppose $p, q\colon F \to G$ are two morphisms of sheaves and take their equalizer $E \xrightarrow{e} F \rightrightarrows G$ as presheaves (since **Set** is complete, so is $[\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}]$, in particular it has equalizers). A basic fact about Hom-functors is that they preserve limits, hence for any presheaf P on X, $\mathrm{Nat}(P, -)$ preserves limits, so the rows of the following diagram are equalizers:

$$\begin{array}{ccccc} \mathrm{Nat}(\mathbf{y}U, E) & \xrightarrow{\mathrm{Nat}(\mathbf{y}U, e)} & \mathrm{Nat}(\mathbf{y}U, F) & \overset{\mathrm{Nat}(\mathbf{y}U,p)}{\underset{\mathrm{Nat}(\mathbf{y}U,q)}{\rightrightarrows}} & \mathrm{Nat}(yU, G) \\ \downarrow{\scriptstyle (i_S)^*_E} & & \downarrow{\scriptstyle (i_S)^*_F}\ \cong & & \downarrow{\scriptstyle (i_S)^*_G}\ \cong \\ \mathrm{Nat}(S, E) & \xrightarrow{\mathrm{Nat}(S, e)} & \mathrm{Nat}(S, F) & \overset{\mathrm{Nat}(S,p)}{\underset{\mathrm{Nat}(S,q)}{\rightrightarrows}} & \mathrm{Nat}(S, G) \end{array}$$

Here, $U \subset X$ is open, S is an arbitrary covering sieve of U, the vertical arrows are induced by the inclusion $i_S\colon S \Rightarrow \mathbf{y}U$, and the diagram commutes (note that with right squares, either pick both top or both bottom arrows). By Theorem 1.30, it is equivalent to show that $(i_S)^*_E$ is an isomorphism; to do so, we construct its inverse by diagram chase, similar as in proof of 5-lemma. Suppose $\eta \in \mathrm{Nat}(S, E)$, so there is a $\rho_\eta \in \mathrm{Nat}(\mathbf{y}U, F)$ such that $(i_S)^*_F(\rho_\eta)$. Now

$$\begin{aligned} (i_S^*)_G\big(\mathrm{Nat}(\mathbf{y}U, p)(\rho_\eta)\big) &= \mathrm{Nat}(S, p)\big(\mathrm{Nat}(S, e)(\eta)\big) \\ &= \mathrm{Nat}(S, q)\big(\mathrm{Nat}(S, e)(\eta)\big) \\ &= (i_S)^*_G\big(\mathrm{Nat}(\mathbf{y}U, q)(\rho_\eta)\big) \end{aligned}$$

where the second equality holds because the bottom row is an equalizer. Since $(i_S)^*_G$ is an isomorphism and the top row is an equalizer, there is a unique $\Theta_\eta \in \mathrm{Nat}(\mathbf{y}U, E)$ such that $\mathrm{Nat}(\mathbf{y}U, e)(\Theta_\eta) = \rho_\eta$. The inverse to $(i_S)^*_E$ is thus given by $\eta \mapsto \Theta_\eta$.

To see $\mathbf{Sh}(X)$ is closed for finite products, note that the product of presheaves F and G is given pointwise, i.e. $(F \times G)(U) = F(U) \times G(U)$ for any open $U \subset X$, and this clearly yields a presheaf. To see that $F \times G$ is a sheaf, we consider the diagram:

$$\begin{array}{ccccc} F(U) & \longleftarrow & F(U) \times G(U) & \longrightarrow & G(U) \\ \downarrow & & \downarrow & & \downarrow \\ \prod_i F(U_i) & \longleftarrow & \prod_i (F \times G)(U_i) & \longrightarrow & \prod_i G(U_i) \\ \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\ \prod_{i,j} F(U_i \cap U_j) & \longleftarrow & \prod_{i,j} (F \times G)(U_i \cap U_j) & \longrightarrow & \prod_{i,j} G(U_i \cap U_j) \end{array}$$

Here, the left and right columns are equalizers, and the horizontal arrows are projections, which satisfy a universal property for products. That the middle column is also an equalizer follows easily from these two properties. The same argument works for arbitrary small products. ■

Corollary 1.33. Any subobject of a sheaf in $\mathbf{Sh}(X)$ is isomorphic to a subsheaf of F.

Proof. Recall that a subobject is an equivalence class of monomorphisms into F; let $m\colon H \Rightarrow F$ be its representative. The map m is a monomorphism if and only if the following square is pullback in $\mathbf{Sh}(X)$:

$$\begin{array}{ccc} H & \xrightarrow{1_H} & H \\ {\scriptstyle 1_H}\Downarrow & \lrcorner & \Downarrow{\scriptstyle m} \\ H & \xrightarrow[m]{} & F \end{array}$$

Since $I\colon \mathbf{Sh}(X) \to [\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}]$ preserves limits, this square is also a pullback in $[\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}]$, but pullbacks in this category are computed pointwise, so m is pointwise monic, i.e. $m_U\colon H(U) \to F(U)$ is monic for any open subset $U \subset X$, which means that H is isomorphic to the subfunctor S of F, given as $S(U) = m_U(H(U))$. ∎

Proposition 1.34. For any space X, $\mathbf{Sh}(X)$ is cocomplete.

Proof. It is enough to show that $\mathbf{Sh}(X)$ is closed for coequalizers and small coproducts (as before, see [1, Chapter V, Section 2, Corollary 2]).

We first show that $\mathbf{Sh}(X)$ is closed for small coproducts. By Corollary 1.18, the sheafification functor $\Gamma\Lambda\colon [\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(X)$ is left adjoint to the inclusion functor, so it preserves colimits, and if F is a sheaf, $\Gamma\Lambda(F) \cong F$. The coproduct $F \coprod G$ (in presheaves) of two sheaves is just a pointwise disjoint union, and this yields a sheaf $\Gamma\Lambda(F \coprod G)$, which must be the coproduct of F and G (in sheaves) since $\Gamma\Lambda$ preserves colimits:

$$F \cong \Gamma\Lambda F \longrightarrow \Gamma\Lambda(F \textstyle\coprod G) \longleftarrow \Gamma\Lambda G \cong G$$

This means that the coproduct of two sheaves is the sheafification of their pointwise disjoint union. A similar proof works for arbitrary small products.

For coequalizers, suppose F and G are sheaves, and let $F \rightrightarrows G \to H$ be a coequalizer diagram in the category of presheaves. Since $\Gamma\Lambda$ preserves colimits, we get that

$$\Gamma\Lambda(F) \cong F \rightrightarrows \Gamma\Lambda(G) \cong G \longrightarrow \Gamma\Lambda(H)$$

is a coequalizer diagram in sheaves. Thus $\Gamma\Lambda(H)$ is the wanted coequalizer. ∎

1.5.2 $\mathbf{Sh}(X)$ is cartesian closed

Recall that a category $\mathbf{C}$ is said to be *cartesian closed*, if:

(i) $\mathbf{C}$ has a terminal object,

(ii) any two objects have a product in $\mathbf{C}$,

(iii) any two objects have an exponential in $\mathbf{C}$.

We have already seen that $\mathbf{y}(X) = \mathrm{Hom}(-, X)$ is the terminal object in $\mathbf{Sh}(X)$, and that $\mathbf{Sh}(X)$ is closed for finite products. That a product of sheaves is in fact just a product of presheaves, suggests that the exponential for presheaves can also be used for sheaves. So we first study at exponentials of presheaves.

Lemma 1.35. For any small category $\mathbf{C}$, the category of presheaves $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ is cartesian closed.

Proof. Recall that the product of two functors $P, Q \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ is their pointwise product. To find the formula for the exponential, we first assume that Q^P exists, so we have $\mathrm{Nat}(R \times P, Q) \cong \mathrm{Nat}(R, Q^P)$ for all presheaves R. In particular, this holds for $R = \mathbf{y}(c) = \mathrm{Hom}_{\mathbf{C}}(-, c)$, and composing this isomorphism with the Yoneda isomorphism yields the isomorphism

$$Q^P(c) \cong \mathrm{Nat}(\mathbf{y}(c), Q^P) \cong \mathrm{Nat}(\mathbf{y}(c) \times P, Q).$$

We now drop the assumption that Q^P exists and use this to define Q^P:

$$Q^P(c) = \mathrm{Nat}(\mathbf{y}(c) \times P, Q),$$

and now $Q^P \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ is clearly a presheaf, as a composition of functors. To show that Q^P is an exponential object, define the evaluation map as

$$\mathrm{eval} \colon Q^P \times P \Rightarrow Q, \quad \mathrm{eval}_c(\eta, y) = \eta_c(1_c, y)$$

for all $c \in \mathbf{C}$, $\eta \colon \mathbf{y}(c) \times P \Rightarrow Q$ and $y \in P(c)$. Naturality of eval is a straightforward computation; hence eval is a morphism of presheaves. Lastly, given a natural transformation $\vartheta \colon R \times P \to Q$, we must provide the transpose map, i.e. a unique map $\theta \colon R \Rightarrow Q^P$ that makes the following diagram commute.

$$\begin{array}{ccc} R \times P & & \\ {\scriptstyle \theta \times 1_P} \Downarrow & \overset{\vartheta}{\searrow} & \\ Q^P \times P & \underset{\mathrm{eval}}{\Longrightarrow} & Q \end{array}$$

This means that we have to define, given $c \in \mathbf{C}$ and $x \in R(c)$, an element $\theta_c(x) \in Q^P(c)$, i.e. a natural transformation $\theta_c(x) \colon \mathbf{y}(c) \times P \Rightarrow Q$. We define

$$(\theta_c(x))_d \colon \mathrm{Hom}_{\mathbf{C}}(d, c) \times P(d) \to Q(d), \quad (\theta_c(x))_d(f, z) = \vartheta_d(R(f)(x), z).$$

The naturality of this map in d follows from naturality of ϑ, and the triangle above commutes since

$$\mathrm{eval}_c(\theta_c(x), y) = (\theta_c(x))_c(1_c, y) = \vartheta_c(x, y).$$

This last computation (together with naturality of θ) also guarantees uniqueness of θ. ∎

Proposition 1.36. If F is a sheaf and P is a presheaf on a space X, then the presheaf F^P is a sheaf.

Proof. Since $\mathscr{O}(X)$ is a small category, F^P is defined (as a presheaf) as in the last lemma as

$$F^P(U) = \mathrm{Nat}(\mathbf{y}(U) \times P, F),$$

where $\mathbf{y}(U) \in [\mathscr{O}(X)^{\mathrm{op}}, \mathbf{Set}]$ is the representable presheaf, given for any $V \in \Omega(X)$ as

$$\mathbf{y}(U)(V) = \mathrm{Hom}(V, U) = \begin{cases} 1 & \text{if } V \subset U, \\ \emptyset & \text{otherwise.} \end{cases}$$

This implies that any $\eta \in F^P(U)$ is determined on all subsets $V \subset U$, since the map

$$\eta_V \colon \mathrm{Hom}(V, U) \times P(V) \to F(V)$$

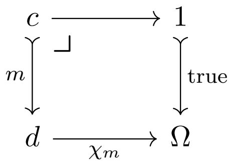
is a trivial map whenever $V \not\subset U$. Hence $F^P(U) \cong \mathrm{Nat}\big(P|_{\mathscr{O}(U)^{\mathrm{op}}}, F|_{\mathscr{O}(U)^{\mathrm{op}}}\big)$, where $P|_{\mathscr{O}(U)^{\mathrm{op}}}$ and $F|_{\mathscr{O}(U)^{\mathrm{op}}}$ denote the restriction of P and F to $\mathscr{O}(U)^{\mathrm{op}}$. Notice that F^P is the functor $\mathscr{O}(X)^{\mathrm{op}} \to$ **Set** which maps $V \hookrightarrow U$ to the map $\mathrm{Nat}\big(P|_{\mathscr{O}(U)^{\mathrm{op}}}, F|_{\mathscr{O}(U)^{\mathrm{op}}}\big) \to \mathrm{Nat}\big(P|_{\mathscr{O}(V)^{\mathrm{op}}}, F|_{\mathscr{O}(V)^{\mathrm{op}}}\big)$, which is just the restriction $\eta \mapsto \eta|_{\mathscr{O}(V)^{\mathrm{op}}}$.

To see that F^P is a sheaf, let $(U_i)_i$ be an open cover of U, and suppose we have elements $\tau_i \in \mathrm{Nat}\big(P|_{\mathscr{O}(U_i)^{\mathrm{op}}}, F|_{\mathscr{O}(U_i)^{\mathrm{op}}}\big)$ with the property that $\tau_i|_{\mathscr{O}(U_i \cap U_j)^{\mathrm{op}}} = \tau_j|_{\mathscr{O}(U_i \cap U_j)^{\mathrm{op}}}$. We can collate these τ_i to a $\tau \in \mathrm{Nat}(P, F)$ in the following way: if $V \subset U$ is any open subset, we define $\tau_V \colon P(V) \to F(V)$ as $f \mapsto \tau_V(f)$, where $\tau_V(f)$ is the collation of elements $\tau_i(f)|_{V \cap U_i}$ – this can be done since $(V \cap U_i)_i$ is an open cover of V, due to the property that τ_i's coincide on intersections $\mathscr{O}(U_i \cap U_j)^{\mathrm{op}}$, and because F is a sheaf. ■

Remark 1.37. The proof of the last proposition shows that, given sheaves F and G, their exponential F^G is the sheaf of germs of morphisms $G \Rightarrow F$.

1.5.3 Sh(X) has a subobject classifier

Recall that a *subobject classifier* for a category **C** (with the terminal object 1) is a pair (Ω, true), where $\Omega \in \mathrm{Obj}(\mathbf{C})$ and $\mathrm{true} \colon 1 \to \Omega$ is a monomorphism in **C**, with the following property: for any monomorphism $m \colon c \rightarrowtail d$, there is a unique *characteristic morphism* $\chi_m \colon d \to \Omega$, i.e. a morphism which makes the following diagram pullback:



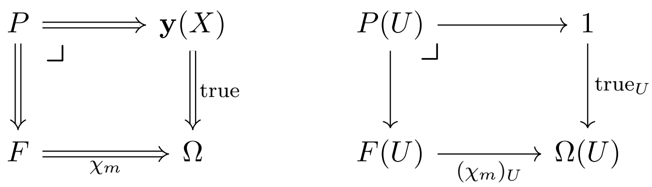
Proposition 1.38. For a topological space X, the subobject classifier for **Sh**(X) is the pair $(\Omega \colon \mathscr{O}(X)^{\mathrm{op}} \to \mathbf{Set}, \mathrm{true} \colon \mathbf{y}(X) \Rightarrow \Omega)$, where:

- $\Omega(U)$ are the open subsets of U, and $\Omega(V \hookrightarrow U) \colon \Omega(U) \to \Omega(V)$ is the restriction to V,
- $\mathrm{true}_U \colon \mathbf{y}(X)_U \to \Omega(U)$ maps $(U \hookrightarrow X) \mapsto U$.

Proof. Let $m \colon S \Rightarrow F$ be any monomorphism of sheaves. By Corollary 1.33, we may assume that S is a subsheaf of F, so that we have $S(U) \subset F(U)$ for any open subset $U \subset X$. Define the promised characteristic morphism $\chi_m \colon F \Rightarrow \Omega$ as

$$(\chi_m)_U \colon F(U) \to \Omega(U),$$
$$(\chi_m)_U(f) = V_{U,f} := \cup\{V \subset U \mid f|_V \in S(V)\},$$

which is clearly a natural transformation, hence a morphism of sheaves. To demystify this definition, we note that in the case $F = C$ and $S = C^1$, $V_{U,f}$ is the largest open subset of U on which the continuous function f is continuously differentiable. By Proposition 1.8 that characterized subsheaves, we have that any $f \in F(U)$ satisfies $f|_{V_{U,f}} \in S(V|_{U,f})$. Now consider the pullback P of true along χ_m:



Here, the left diagram is in $\mathbf{Sh}(X)$, while the right one is in **Set**. This pullback in **Set** is just

$$P(U) = \{f \in F(U) \mid (\chi_m)_U(f) = U\} = \{f \in F(U) \mid V_{U,f} = U\} = S(U),$$

where the last equality follows from $f|_{V_{U,f}} \in S(V|_{U,f})$. In other words, the pullback P is exactly the subsheaf S.

To show uniqueness of χ_m, suppose S is the pullback of true along some other morphism $\psi\colon F \Rightarrow \Omega$. This means $S(U) = \{f \in F(U) \mid \psi_U(f) = U\}$, and it is clear that for $S(U)$ as above, this implies $\psi_U(f) = V_{U,f}$, so $\psi = \chi_m$. ■

2 Brief introduction to topos theory

This section serves to give elementary definitions of sheaves on Grothendieck sites, so we will keep the upcoming discussion relaxed and brief. Let $\mathbf{C}$ denote an arbitrary category.

2.1 Grothendieck topologies on categories

The task at hand is to introduce the notion of a sheaf on $\mathbf{C}$. First off, a *presheaf* on $\mathbf{C}$ is defined as a functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, so that the category of presheaves $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ on $\mathbf{C}$ is just a functor category, as before.

To generalize sheaves from a topological space to an arbitrary category, we notice that by Theorem 1.30 we have to introduce the notion of a covering sieve on an object c of $\mathbf{C}$ (we already know what a sieve on c is, see Definition 1.27 and Proposition 1.28). Consider the case $\mathbf{C} = \mathscr{O}(X)$ for a given topological space X; the properties of the set $J(U)$ of all covering sieves on an open subset $U \subset X$ will provide the means of axiomatization of covering sieves on objects of arbitrary categories. In the spirit of category theory, we consider morphisms of $\mathscr{O}(X)$, so let $i_{V,U}\colon V \to U$ be any inclusion of open subsets in X, and define the *pullback of a sieve* S from U to V as

$$(i_{V,U})^*(S) = \{W \in S \mid W \subset V\} = \{W \cap V \mid W \in S\},$$

which is clearly a sieve on V. The important properties of $J(U)$ are the following:

(i) $\Omega(U)$ is a covering sieve, i.e. $\Omega(U) \in J(U)$.

(ii) Pullbacks preserve covering sieves, i.e. if $S \in J(U)$, then $(i_{V,U})^*(S) \in J(V)$.

(iii) If S is a covering sieve, and R is a sieve on U such that the sieve $(i_{V,U})^*(R)$ covers U for any $V \in S$, then R is a covering sieve as well.

It is a triviality that these properties indeed hold for the set $J(U)$ of all covering sieves on U. This now enables us to define covering sieves of an arbitrary category $\mathbf{C}$. Before doing so, we define the *pullback of a sieve* S on c along an arrow $f\colon d \to c$ in $\mathbf{C}$ as

$$f^*(S) = \{g \mid \mathrm{cod}(g) = d, fg \in S\},$$

which is evidently a sieve on d.

Definition 2.1. A *(Grothendieck) topology* on a category $\mathbf{C}$ is a map J which assigns to any object c a collection $J(c)$ of sieves on c, such that the following holds.

(i) The *maximal sieve* $t_c = \{g \mid \mathrm{cod}(g) = c\}$ is in $J(c)$.

(ii) (*Stability*) Pullbacks preserve covering sieves, i.e. if $S \in J(c)$, then $f^*(S) \in J(d)$ for any arrow $f\colon d \to c$.

(iii) (*Transitivity*) If $S \in J(c)$, and R is a sieve on c such that $f^*(R) \in J(d)$ for any arrow $f \in S$ with domain d, then $R \in J(c)$.

A *site* on $\mathbf{C}$ is a pair $(\mathbf{C}, J)$, i.e. a category $\mathbf{C}$ endowed with a topology J. For any object c in $\mathbf{C}$, elements of $J(c)$ are called *covering sieves* of c, and we say that elements $S \in J(c)$ *cover the object* c. Furthermore, we say that a sieve S on c *covers an arrow* $f\colon d \to c$, if $f^*(S) \in J(d)$.

Remark 2.2. Any Grothendieck topology is *upwards closed*, i.e. if $S \in J(c)$ then any larger sieve $R \supset S$ is also in $J(c)$. To show this using the transitivity axiom, take any $f\colon d \to c$ from S and note that $\mathrm{id}_d \in f^*(S)$, hence $f^*(S)$ must be the maximal sieve on d, but $f^*(S) \subset f^*(R)$, hence $f^*(R)$ is also the maximal sieve on d, so $f^*(R) \in J(d)$.

Trivially, a sieve S covers c if and only if S covers id_c. In the language of sieves covering arrows, the axioms read:

(ia) For any sieve S on c and $f \in S$, S covers f.

(iia) If S covers an arrow $f\colon d \to c$, it covers the composition $f \circ g$, for any arrow $g\colon e \to d$.

(iiia) If S covers an arrow $f\colon d \to c$ and R is a sieve on c covering all arrows of S, R covers f

For example, to show (ia) follows from (i), we have to check that $f^*(S)$ is a covering sieve on d for any $f \in S$ with $f\colon d \to c$, but since $fg \in S$ for all g whenever the composition is defined, we must have $f^*(S) = t_d \in J(d)$ by (i). For the converse implication, we have by assumption that the maximal sieve t_c covers $f = \mathrm{id}_c \in t_c$, hence $t_c = \mathrm{id}_c^*(t_c) \in J(c)$. We leave it to the reader to check the equivalences (ii)$\leftrightarrow$(iia) and (iii)$\leftrightarrow$(iiia).

We also note that if $R, S \in J(c)$, then $R \cap S \in J(c)$. Indeed, for any $f \in R$, we must have $f^*(R \cap S) = f^*(S) \in J(c)$ by (ii), hence $R \cap S \in J(c)$ by (iii).

2.2 Sheaves on sites

Mimicking the content of Theorem 1.30, we can now define sheaves on sites. Recall from Proposition 1.28 that any sieve S on c can be thought of as a subfunctor of $\mathbf{y}(c)$.

Definition 2.3. A *sheaf* on a site $(\mathbf{C}, J)$ is a presheaf $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, such that for every object c in $\mathbf{C}$ and any covering sieve $S \in J(c)$, the inclusion $S \Rightarrow \mathbf{y}(c)$ induces an isomorphism $\mathrm{Nat}(S, F) \cong \mathrm{Nat}(\mathbf{y}(c), F)$.

We can provide a more intuitive way of thinking about sheaves on sites. The following definitions mimic the topological setting of a family of functions that match on intersections, and their collation to a global function.

Definition 2.4. Let P be a presheaf on a site $(\mathbf{C}, J)$ and let a sieve $S \in J(c)$ cover an object c. A *matching family* for S is a family of elements $(x_f)_{f \in S}$ where $x_f \in P(\mathrm{dom}(f))$, such that for any $g\colon e \to d$ in $\mathbf{C}$, there holds

$$x_f \cdot g := P(g)(x_f) = x_{fg}.$$

(Note that $fg \in S$, because S is a sieve.) An *amalgamation* of a matching family x for a covering sieve S of c is an element $x \in P(c)$, such that for any $f \in S$, there holds

$$x \cdot f := P(f)(x) = x_f.$$

Remark 2.5. Just as a sieve S on c can be seen as a subfunctor of $\mathbf{y}(c)$, a matching family $(x_f)_{f\in S}$ can be seen as a natural transformation $\Theta\colon S \Rightarrow P$ (defined as $\Theta_d(f) = x_f$, where $d = \mathrm{dom}(f)$). This means that in a presheaf, every matching family for any cover of any object has a unique amalgamation, if and only if for all covering sieves S of objects c, any natural transformation $\Theta\colon S \Rightarrow P$ has a unique extension to $\mathbf{y}(c)$, that is:

$$\begin{array}{ccc} S & \xRightarrow{\ \Theta\ } & P \\ \Downarrow & \nearrow & \\ \mathbf{y}(c) & & \end{array}$$

This means that a presheaf P is a sheaf if and only if every matching family for any cover of any object has a unique amalgamation. The latter can also be expressed diagrammatically, by requiring that for any object c in $\mathbf{C}$ and each covering sieve S of c, the following diagram is equalizer in $\mathbf{Set}$:

$$P(c) \xrightarrow{\ e\ } \prod_{f\in S} \underset{a}{\overset{p}{\rightrightarrows}} \prod_{\substack{f,g \ f\in S,\\ \mathrm{dom}\, f = \mathrm{cod}\, g}} P(\mathrm{dom}\, g)$$

Here, the maps are given as

$$e(x) = (x \cdot f)_{f\in S}, \quad p(x_f)_{f\in S} = (x_{fg})_{f,g}, \quad a(x_f)_{f\in S} = (x_f \cdot g)_{f,g}.$$

Remark 2.6. The sheaves on $(\mathbf{C}, J)$ form a category $\mathbf{Sh}(\mathbf{C}, J)$, where the maps are just natural transformations, so that this is a full subcategory of $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$, and we again have the inclusion functor $I\colon \mathbf{Sh}(\mathbf{C}, J) \to [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$. Any category, equivalent to the category $\mathbf{Sh}(\mathbf{C}, J)$ of sheaves on some site, is called a *Grothendieck topos*.

Similarly to the case $\mathbf{C} = \mathscr{O}(X)$, we have the following theorem.

Theorem 2.7. The inclusion functor $I\colon \mathbf{Sh}(\mathbf{C}, J) \to [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ has a left adjoint

$$\mathbf{a}\colon [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathbf{C}, J),$$

called the *associated sheaf functor*. The composition $\mathbf{a} \circ I\colon \mathbf{Sh}(\mathbf{C}, J) \to \mathbf{Sh}(\mathbf{C}, J)$ is naturally isomorphic to the identity functor.

The construction of the left adjoint functor $\mathbf{a}$ of I is beyond the scope of this exposition; so is the fact that $\mathbf{Sh}(\mathbf{C}, J)$ is an elementary topos. For the construction of the adjoint functor $\mathbf{a}$, see [2, Chapter III, Section 5], and for the construction of subobject classifier, see [2, Chapter III, Section 7].

References

[1] Saunders MacLane. *Categories for the Working Mathematician.* Springer, 2013.

[2] Saunders MacLane and Ieke Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory.* Springer, 2012.