Yang–Mills theory for multiplicative Ehresmann connections

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Short abstract

Yang–Mills theory is a framework for finding connections with minimal curvature, using variational calculus. This poster is about a generalization of Yang–Mills theory, from principal bundles to possibly *non-transitive* Lie groupoids and algebroids. In our framework, principal bundle connections are replaced with the more general notion of *(infinitesimal) multiplicative Ehresmann connections*, which are a type of connections compatible with the structure of a Lie groupoid (or algebroid), introduced by Fernandes & Mărcuţ.

Our action functional for such connections includes a *curvature 3-form* contribution, alongside the usual curvature 2-form term; the resulting variational problem is naturally constrained by a cohomological condition. We present the associated Euler–Lagrange equations, demonstrate the gauge invariance of the solution space, and relate the global and infinitesimal theory via the van Est map. As an important application, we show that our framework produces a Yang–Mills theory for bundle gerbes.

Classical theory from a Lie algebroid viewpoint

In the work of Atiyah and Bott (1983), it was already recognized that connections on a principal G-bundle $\pi\colon P \to M$ can equivalently be described as vector bundle splittings of the *Atiyah sequence*,

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{ad}(P) \arrow[r] & A(P) \arrow[r, "\rho"'] & TM \arrow[l, bend right, "\sigma"'] \arrow[r] & 0. \end{tikzcd} \tag{1}$$

Here, $A(P) = \frac{TP}{G}$ is the Atiyah algebroid of $P \to M$, $\mathrm{ad}(P) = \frac{P \times \mathfrak{g}}{G}$ is the adjoint vector bundle, and ρ is induced by $\mathrm{d}\pi$. The interpretation is that a splitting σ is the same as a horizontal lift of vector fields on M, to G-invariant ones on P. Since (1) is a short exact sequence of algebroids, any splitting σ has a *curvature* F^σ, measuring its failure to be a Lie algebroid morphism,

$$F^\sigma \in \Omega^2(M; \mathrm{ad}\, P), \quad F^\sigma(X,Y) = \sigma([X,Y]) - [\sigma(X), \sigma(Y)].$$

A splitting σ also induces a connection ∇^σ on $\mathrm{ad}(P)$, given by $\nabla^\sigma_X \xi = [\sigma(X), \xi]$. The Jacobi identity on the Atiyah algebroid $A(P)$ implies that the triple $(\sigma, \nabla^\sigma, F^\sigma)$ satisfies:

(i) ∇^σ preserves the Lie bracket on $\mathrm{ad}(P)$.
(ii) The curvature R^{∇^σ} of ∇^σ is given by the adjoint action by F^σ, i.e., $R^{\nabla^\sigma} = [-, F^\sigma]$,
(iii) The curvature satisfies the Bianchi identity: $\mathrm{d}^{\nabla^\sigma} F^\sigma = 0$.

To formulate a variational problem, we need to additionally pick a metric on the (oriented) base manifold M, and an ad-invariant metric on $\mathrm{ad}(P)$. We can then define the so-called *Yang–Mills action functional*,

$$\mathscr{S}(\sigma) = \int_M \langle F^\sigma, F^\sigma \rangle \, \mathrm{vol}_M.$$

Its critical points are precisely the splittings σ whose curvature satisfies the *Yang–Mills equation*,

$$\mathrm{d}^{\nabla^\sigma} \star F^\sigma = 0.$$

Notice that this construction does not depend on global data! This offers an important insight: "Yang–Mills theory is fundamentally *infinitesimal in nature*". Let us now describe the non-transitive case.

What is our setting?

Instead of the Atiyah sequence (1), we start with any short exact sequence of Lie algebroids,

$$0 \longrightarrow \mathfrak{k} \hookrightarrow A \xrightarrow{\phi} B \longrightarrow 0, \tag{2}$$

living over the same base manifold M. Equivalently, we can start with a ***bundle of ideals*** $\mathfrak{k} \subset A$, which is a vector subbundle contained in the isotropy of A, whose sections $\Gamma(\mathfrak{k})$ form an ideal for the Lie bracket on A,

$$[\Gamma(A), \Gamma(\mathfrak{k})] \subset \Gamma(\mathfrak{k}).$$

This just means $\mathfrak{k}$ is a representation of A, via the adjoint action, and we have $B = A/\mathfrak{k}$ (ϕ is the projection). The Lie algebroid A is called an *extension* of the Lie algebroid B by a bundle of Lie algebras $\mathfrak{k}$, and both algebroids A and B determine the same (singular) orbit foliation on M, denoted $\mathcal{F}$.

What are our fields?

In the classical case, the dynamical fields (i.e., the variables that the action functional takes as inputs) were simply the splittings of (1). For general Lie algebroid extensions, the natural connections are no longer just splittings $\sigma\colon B \to A$ of (2), but now contain additional data. This data naturally comprises a *multiplicative Ehresmann connection*. We can think of them in a down-to-earth way: as triples (σ, ∇, F), satisfying the compatibility relations below, extending the conditions (i–iii) we saw in the transitive case.

Definition. Given a Lie algebroid extension as above, a ***connection triple*** (σ, ∇, F) consists of a vector bundle splitting $\sigma\colon B \to A$ of the sequence (2), a linear connection ∇ on $\mathfrak{k}$, and a 2-form $F \in \Omega^2(M; \mathfrak{k})$. They must satisfy the following compatibility conditions:

(i) ∇ preserves the Lie bracket on $\mathfrak{k}$.
(ii) $R^\nabla = [-, F]_{\mathfrak{k}}$.
(iii) The *curvature 3-form* $G \coloneqq \mathrm{d}^\nabla F$ is transversal: $\iota_X G = 0$ for all $X \in T\mathcal{F}$.
(iv) $\nabla_{\rho_B(\beta)} = [\sigma(\beta), -]$, for any $\beta \in B$.
(v) $F^\sigma = (\rho_B)^* F$.

"Leafwise (tangentially to $\mathcal{F}$), ∇ and F are completely determined by σ."

As before, F^σ is the curvature of the splitting σ, measuring its failure to be an algebroid morphism:

$$F^\sigma \in \Omega^2(B; \mathfrak{k}), \quad F^\sigma(\beta_1, \beta_2) = \sigma([\beta_1, \beta_2]) - [\sigma(\beta_1), \sigma(\beta_2)].$$

Ever heard about $\mathcal{VB}$-algebroids, infinitesimally multiplicative (IM) forms, and the Weil complex?

A triple as above can alternatively be described as a pair (ω, F), where ω is a MEC, and F is its *curving*. More precisely, a ***MEC*** for $\phi\colon A \to B$ is defined as either of the following two equivalent things:

- A distribution $E \subset TA$ which is a $\mathcal{VB}$-subalgebroid of $TA \Rightarrow TM$, satisfying $TA = E \oplus \ker \mathrm{d}\phi$.
- An infinitesimally multiplicative (IM) 1-form $\omega \in \Omega^1_{im}(A; \mathfrak{k})$ with values in the representation $\mathfrak{k}$, whose symbol is a vector bundle splitting of the short exact sequence (2).

The 2-form $F \in \Omega^2(M; \mathfrak{k})$ plays the role of a "potential", modelling an abstract object — the abstract curvature $\Omega^\omega \in \Omega^2_{im}(A; \mathfrak{k})$, which measures the failure of involutivity of E w.r.t. $TA \Rightarrow A$ (!). The precise relationship to F is: the curvature Ω^ω is exact in the *Weil complex* (take form degree $k = 2$),

$$W^{0,k}(A; \mathfrak{k}) = \Omega^k(M; \mathfrak{k}) \xrightarrow{\delta^0} \Omega^k_{im}(A; \mathfrak{k}) \subset W^{1,k}(A; \mathfrak{k}) \xrightarrow{\delta^1} W^{2,k}(A; \mathfrak{k}) \xrightarrow{\delta^2} \cdots \tag{3}$$

and F is its primitive, $\Omega^\omega = \delta^0 F$. (Note that IM forms are 1-cocycles, i.e., $\Omega^\bullet_{im}(A; \mathfrak{k}) = \ker \delta^1$.)

The ***Bianchi identity*** in condition (iii) no longer reads $\mathrm{d}^\nabla F = 0$, as opposed to the transitive case! Instead, condition (iii) is equivalent to saying that *the curvature 3-form G is an A-invariant form*,

$$\delta^0 G = 0.$$

Roughly, this means we regard G as a form on the (possibly non-smooth) leaf space $M/\mathcal{F}$. This leads to a conceptual shift about the curvature of a connection triple: it is now encoded by a *curvature pair* (F, G)!

The Euler–Lagrange equations

When constructing an action functional for connection triples, the idea is that besides accounting for the norm of the 2-curvature, the norm of the corresponding 3-curvature should also contribute to the action.

Main theorem. *Let $\mathfrak{k}$ be a bundle of ideals of a Lie algebroid A over an oriented pseudo-Riemannian manifold M, and let $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ be an ad-invariant bundle metric on $\mathfrak{k}$. For any connection triple (σ, ∇, F), the following statements are equivalent.*

*(i) The triple (σ, ∇, F) is a solution to the **constrained variational problem** for the action*

$$\mathscr{S}(\sigma, \nabla, F) = \int_M \langle F, F \rangle_{\mathfrak{k}} \, \mathrm{vol}_M + \mu \int_M \langle G, G \rangle_{\mathfrak{k}} \, \mathrm{vol}_M, \quad (\mu \in \mathbb{R}), \tag{A}$$

subject to the constraint that the cohomological class of the underlying MEC is constant in the Weil cohomology of A: $[\omega] = const. \in H^1(W^{\bullet,1}(A; \mathfrak{k}), \delta)$.

(ii) The curvature pair (F, G) satisfies the following Euler–Lagrange equation and relation,

$$(\mathrm{d}^\nabla)^* F = 0, \tag{YM.I}$$

$$F + \mu\, (\mathrm{d}^\nabla)^* G \perp \Omega^2_{A\text{-inv}}(M; \mathfrak{k}), \tag{YM.II}$$

which are systems of nonlinear partial differential equations and relations (PDR), respectively.

Here, the space $\Omega^\bullet_{A\text{-inv}}(M; \mathfrak{k}) = \ker \delta^0$ denotes *A-invariant forms*, which model the forms on the (singular) leaf space $M/\mathcal{F}$. The differential operator $(\mathrm{d}^\nabla)^*$ denotes the covariant coderivative, i.e., the formal adjoint to d^∇ with respect to the L^2-pseudo inner product on $\Omega^\bullet(M; \mathfrak{k})$, induced by integrating over M.

Gauge invariance of solution space

Given a variational problem, *gauge invariance* refers to the principle that "any automorphism which preserves the underlying structure defining the problem, should also preserve its solution space." More precisely:

Definition. Assume the setting of Main theorem. An ***infinitesimal gauge transformation*** is a Lie algebroid automorphism Θ of $A \Rightarrow M$ covering an orientation-preserving isometry on the base M, restricting on $\mathfrak{k} \subset A$ to an isometry for the bundle metric $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$.

Any such Θ is in particular an automorphism of the extension (2), since we are requiring $\Theta(\mathfrak{k}) = \mathfrak{k}$. Gauge transformations thus naturally act on connection triples by pullbacks.

Theorem. *In the setting of Main theorem, the action functional (A) is invariant under infinitesimal gauge transformations. Hence, the solution space to (YM.I) and (YM.II) is preserved by them.*

Relating back to principal bundles: for an integrable algebroid A, a class of gauge transformations is given by adjoint actions of (static) bisections of an integrating (source-fibre connected) Lie groupoid $\mathcal{G}$. These are differentials $\mathrm{Ad}_\sigma = (I_\sigma)_*\colon A \to A$ of inner automorphisms $I_\sigma\colon \mathcal{G} \to \mathcal{G}$, where σ is a bisection of $\mathcal{G}$. Note that the usual principal bundle automorphisms are precisely the static bisections of its gauge groupoid.

Application: Yang–Mills theory for bundle gerbes

We now showcase the groupoid version of Main theorem, for a particular class of groupoid extensions: bundle gerbes (over a manifold N). These are geometric representatives of $H^3(N; \mathbb{Z})$, similarly as line bundles represent $H^2(N; \mathbb{Z})$. More precisely, a *bundle gerbe over N* is a Lie groupoid $\mathcal{G} \rightrightarrows M$ with orbit space N, whose isotropy is the trivial bundle S^1_M, on which $\mathcal{G}$ acts trivially by conjugation: $C_g(s(g), z) = (t(g), z)$.

$$1 \longrightarrow S^1_M \longrightarrow \mathcal{G} \xrightarrow{(t,s)} M \times_N M \longrightarrow 1. \tag{4}$$

In other words, a bundle gerbe is a *central extension* of a submersion groupoid, by the trivial bundle S^1_M.

The natural connections on a bundle gerbe are precisely MECs for (4), i.e., multiplicative forms $\omega \in \Omega^1_m(\mathcal{G})$ satisfying $\omega|_{\ker(\rho)} = \mathrm{id}$. Since the groupoid cohomology of the submersion groupoid is acyclic, we have:

- Every MEC ω admits a (non-unique) curving $F \in \Omega^2(M)$, that is, $\mathrm{d}\omega = \Omega^\omega = \delta^0 F = s^*F - t^*F$.
- The variational constraint $[\omega] = const. \in H^1(\Omega^1(G^{(\bullet)}), \delta)$ is vacuous. Therefore:

Theorem. *Let $\mathcal{G}$ be a bundle gerbe as above, where M is an oriented, pseudo-Riemannian manifold. A pair (ω, F) is a critical point for the action functional*

$$\mathscr{S}(\omega, F) = \int_M F \wedge \star F + \mu \int_M G \wedge \star G,$$

if and only if its curvature pair $(F, G = \mathrm{d}F)$ satisfies the Euler–Lagrange conditions

$$\mathrm{d}^* F = 0,$$

$$F + \mu\, \mathrm{d}^* G \perp \pi^* \Omega^2(N),$$

where the covariant coderivative reads $\mathrm{d}^* = (-1)^k \star^{-1} \mathrm{d} \star$ on forms of degree k.

Example. Let us consider a *trivial* bundle gerbe over N, i.e., the fibred product of the submersion groupoid of $\pi\colon M \to N$ with the gauge groupoid of an S^1-principal bundle $p\colon P \to M$,

$$\mathcal{G} = \mathcal{G}(P) \times_M (M \times_\pi M) = \{([u_2, u_1], (y, x)) \in \mathcal{G}(P) \times (M \times_\pi M) \mid p(u_1) = x, p(u_2) = y\}.$$

Specifically, take $\pi\colon T^4 \to T^3$, given in standard coordinates by $(x, y, z, t) \mapsto (x, y, z)$, and consider the trivial bundle $P = M \times S^1$. In this case, (4) reduces to a sequence of tori over the base manifold $M = T^4$,

$$1 \longrightarrow T^5 \longrightarrow T^6 \longrightarrow T^5 \longrightarrow 1. \tag{5}$$

Denoting the coordinates on T^6 by $(x, y, z, t_2, t_1, \varphi)$, define a MEC ω and a 2-form F by

$$\omega = \mathrm{d}\varphi + (t_2 - t_1)\, \mathrm{d}x \in \Omega^1(T^6), \qquad F = \cos(x)\, \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}t \wedge \mathrm{d}x \in \Omega^2(M).$$

Since the abstract curvature of ω is $\Omega^\omega = \mathrm{d}\omega = s^*F - t^*F = \delta^0 F$, the form F is a curving of ω, and the 3-curvature equals $G = -\sin(x)\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$. Now, the curvature pair (F, G) satisfies the Euler–Lagrange equations above for the constant $\mu = -1$, since F is a sum of:

(i) a coclosed basic part which is an eigenform of $\mathrm{d}^*\mathrm{d}$, $F_{\mathrm{bas}} = \cos(x)\, \mathrm{d}y \wedge \mathrm{d}z$ (with eigenvalue 1),
(ii) a harmonic vertical part, $F_{\mathrm{vert}} = \mathrm{d}t \wedge \mathrm{d}x$.

Hence, the pair (ω, F) is a solution of the Yang–Mills variational problem for the bundle gerbe (5).

Good questions to ask me (i.e., those which I can answer)

- How are the global and infinitesimal Yang–Mills theory related? Hint: by the *van Est map*,
$$\mathscr{VE}\colon \Omega^k(\mathcal{G}^{(\bullet)}; \mathfrak{k}) \to W^{\bullet,k}(A; \mathfrak{k}).$$
- Is there a notion of self-dual solutions ("**instantons**") for the variational problem for MECs?
- What does the Main theorem yield for the transitive case of the Almeida–Molino algebroid?
- At the other extreme, what happens for the totally intransitive case of Lie algebra bundles?
- What happens when the bundle of ideals $\mathfrak{k}$ has a semisimple typical fibre?
- Do infinitesimal MECs have a supergeometric interpretation, in terms of dg-manifolds?