Lie groupoids: Lecture 8

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III. Connections

Lie algebroids give us a possible generalization of the usual notion of the affine connection on a vector bundle:

Def. Let $A \to M$ be a Lie algbd and $E \to M$ a v.b.
An <u>(affine) A-connection on E</u> is a bilinear map

$$\Gamma^\infty(A) \times \Gamma^\infty(E) \to \Gamma^\infty(E)$$
$$(\alpha, s) \mapsto \nabla_\alpha s$$

which is $C^\infty(M)$-linear in the first argument, and satisfies the Leibniz rule:

$$\nabla_\alpha (fs) = f \nabla_\alpha s + \rho(\alpha)(f)\cdot s.$$

Its <u>curvature</u> is the map

$$R_\nabla : \Gamma^\infty(A) \times \Gamma^\infty(A) \times \Gamma^\infty(E) \to \Gamma^\infty(E)$$
$$R_\nabla(\alpha, \gamma)s = \nabla_\alpha \nabla_\gamma s - \nabla_\gamma \nabla_\alpha s - \nabla_{[\alpha,\gamma]} s.$$

Remark. Clearly, $R_\nabla(\gamma, \alpha)s = -R_\nabla(\alpha,\gamma)s$. Further, R_∇ is $C^\infty(M)$-linear in all three arguments (the proof is the same as in the standard case).
Hence $R_\nabla \in \Gamma^\infty(\Lambda^2(A^*) \otimes \mathrm{End}(E))$ is a tensor field.

Ex. A TM-connection is just the ordinary affine connection.

Remark. Locally (in a frame $(s_i)_{i=1}^r$ of E on U; $r = \mathrm{rank}(E)$)

$$\nabla_\alpha s_i = \sum_{j=1}^r \omega^j{}_i(\alpha) s_j, \quad R(\alpha,\gamma)s_i = \sum_j \Omega^j{}_i(\alpha,\gamma) s_j$$

where $\omega^j{}_i \in \Gamma_U^\infty(A^*)$, $\Omega^j{}_i \in \Gamma_U^\infty(\Lambda^2(A^*))$ are some where $\omega^i_j \in \Gamma^\infty_U(A^*)$, $\Omega^i_j \in \Gamma^\infty_U(\Lambda^2(A^*))$ are some locally defined "differential forms on A". The relation

$$\Omega^i_j = d\omega^i_j + \sum_{k=1}^r \omega^i_k \wedge \omega^k_j$$ STRUCTURE EQUATION

is proved in the same way as for the standard case.

<u>Remark</u>. $\nabla_\alpha s = \nabla_\alpha (f^i s_i) = \rho\alpha(f^i) s_i + f^i \omega^j_i(\alpha) s_j$ from which it's clear that $\nabla_\alpha s|_x$ depends only on α_x and the value of s on an arbitrarily small integral path of $\rho\alpha$ through x.

<u>Parallel transport</u>

We can now generalize the cov. derivative along a path. If $a : I \to A$ is a path in A, in accord with the previous remark we can only covariantly differentiate paths $s : I \to E$ with $\pi_E \circ s = \pi_A \circ a$ and we must have $\rho(a(t)) = \frac{d}{dt}(\pi_A \circ a)$ so that $\pi_A \circ a$ is the integral path of ρa.

Let's summarize this:

<u>Def</u>. An <u>A-path</u> is a map $a : I \to A$ s.t.

$$\rho(a(t)) = \frac{d}{dt}\big(\underbrace{\pi_A(a(t))}_{=:\gamma_a}\big) =: \gamma_a'(t).$$

Now let ∇ be an A-connection on E, and let $s : I \to M$ be a section of E along γ_a, i.e. $\pi_E \circ s = \gamma_a$. The <u>A-derivative of s along a</u> is $D_a s : I \to E$,

$$(D_a s)(t) := \dot f^i(t)\, s_i|_{\gamma_a(t)} + f^i(t)\, \omega^j{}_i(a(t))\, s_j|_{\gamma_a(t)}$$

where we've written $s(t) = f^i(t)\, s_i|_{\gamma_a(t)}$, $f^i : I \to \mathbb{R}$.

Rmk. $a : I \to A$ is an A-path iff $a\,dt : TI \to A$ is a Lie algebra morphism:

$$\begin{tikzcd} I\times\mathbb{R} = TI \arrow[r, "a\,dt"] \arrow[d, "pr_1"'] & A \arrow[d, "\pi_A"] \\ I \arrow[r, "\pi_A \circ a"'] & M \end{tikzcd} \qquad \begin{tikzcd} TI \arrow[r, "a\,dt"] \arrow[d, "id"'] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[d, "\rho"] \\ TI \arrow[r, "d\pi_A \circ da"'] & TM \end{tikzcd} \quad \ldots$$

use:
$a(t) = a\,dt(1_t)$
and
$da(1_t) = a'(t)$

Rmk. Independence of the choice of local frame?

$s_i = A^j{}_i\, \tilde s_j$ for a matrix $A = [A^j{}_i]_{j,i}$ of smooth fn's.

$$\omega^j{}_i = (A^{-1})^j{}_k\, \tilde\omega^k{}_l\, A^l{}_i + (A^{-1})^j{}_k\, dA^k{}_i \circ \rho$$

$f^i(t) = (A^{-1})^i{}_j(\gamma_a(t))\, \tilde f^j(t)$, so

$$\dot f^i(t) = (A^{-1})^i{}_j(\gamma_a(t))\, \dot{\tilde f}^j(t) - dA^i{}_j(\gamma_a'(t))\, \tilde f^j(t)$$

Plug this into the above expression (HW).

Ex.

(i) If $A = TM$, a TM-path is just the speed of a path in M.

(ii) If $A = \frac{TP}{G}$ is the Atiyah algebroid of $\pi : P \xrightarrow{G} M$, what is the requirement that $a : I \to A$ is an A-path?

$a(t) = [v(t)|_{\gamma(t)}]$ for $v : I \to TP$, $\gamma : I \to P$ s.t. $v(t) \in T_{\gamma(t)}P$

The req. is: $\rho(a(t)) := d\pi(v(t)) \overset{\text{req.}}{=} \frac{d}{dt}(\pi(\gamma(t)))$, i.e.

$v(t) - \gamma'(t) \in \ker d\pi_{\gamma(t)} = VP_{\gamma(t)}\ \forall t \in I.$

Prop. Let ∇ be an A-connection on E and $a: I \to A$ an A-path. Given $v \in E_{\gamma_a(t_0)}$ for some $t_0 \in I$, there is a unique parallel section $s: I \to E$ along γ_a s.t. $s(t_0) = v$.

Pf. Let $I = [a,b]$. By Lebesgue's lemma, there is a division $a = t_0 < \dots < t_\ell = b$ s.t. $\gamma_a([t_{i-1}, t_i])$ is contained in a chart nbd. $U_i \subset M$. Since $E|_{U_i}$ is trivial, take a local frame $(s_j^{(i)})_j$ of E on U_i. The solution of the linear system

$$\dot{f}^j_{(i)}(t) + f^k_{(i)}(t)\, \omega^j{}_k(a(t)) = 0 \quad \forall k$$

is then obtained inductively, for all $t \in [t_{i-1}, t_i]$, $i = 1 \dots \ell$. Picard's thm. ensures that the local definition of s

$$s(t) = f^j_{(i)}\, s_j^{(i)}(\gamma_a(t)) \quad \text{if } t \in [t_{i-1}, t_i]$$

patches together smoothly. $\square$

So, given any A-path $a: [0,1] \to A$, we obtain a linear iso.

$$\tau_a : E_{\gamma_a(0)} \to E_{\gamma_a(1)}, \quad \tau_a(v) = s^v(1) \text{ where}$$

s^v is the parallel section of E along γ_a with $s^v(0) = v$.

This is called the <u>parallel transport</u> along the A-path a.

It satisfies:

(i) invariance under reparametrization of paths

(ii) $\tau_{a_1 * a_2} = \tau_{a_1} \circ \tau_{a_2}$

(iii) invariance under homotopy, if $R_\nabla = 0$.

Two A-paths a_0, a_1 are A-path homotopic, if there is a Lie algbd. morphism:

$$\begin{array}{ccc} T(I\times I) & \xrightarrow{\Phi} & A \\ \downarrow & & \downarrow \\ I\times I & \xrightarrow{\phi} & M \end{array}$$

$\Phi = \Phi_1\, dt + \Phi_2\, d\varepsilon$ where

$\Phi_1(t,0) = a_0(t)$, $\Phi_2(t,1) = a_1(t)$

and $\Phi_1(0,\varepsilon) = \Phi_2(0,\varepsilon) = 0$

For proof of (iii), see lecture 11 by Rui L. Fernandes.

Levi-Civita connections

Of special importance are A-connections on A, i.e.

$$\nabla : \Gamma^\infty(A) \times \Gamma^\infty(A) \to \Gamma^\infty(A)$$
$$(\alpha, \eta) \mapsto \nabla_\alpha \eta .$$

Def. The torsion of an A-connection ∇ on A is:

$$T_\nabla : \Gamma^\infty(A) \times \Gamma^\infty(A) \to \Gamma^\infty(A)$$
$$T_\nabla(\alpha,\eta) = \nabla_\alpha \eta - \nabla_\eta \alpha - [\alpha,\eta]$$

Rmk. We again have that T_∇ is antisymmetric, and $C^\infty(M)$-linear:

$$T_\nabla(\alpha, f\eta) = \nabla_\alpha f\eta - \nabla_{f\eta}\alpha - [\alpha, f\eta] =$$
$$= \cancel{\rho\alpha(f)(\eta)} - f\nabla_\eta \alpha - \cancel{\rho\alpha(f)}\cdot\eta - f[\alpha,\eta]$$

Hence $T_\nabla \in \Gamma^\infty(\Lambda^2 (A^*) \otimes A)$.

Ex. obtaining A-connections from a TM-connection ∇ on A:

(i) $\tilde\nabla_\alpha \eta = \nabla_{\rho\alpha}\eta$ ... A-connection on A

$\hookrightarrow$ $\tilde\nabla_\alpha f\eta = \rho\alpha(f)\cdot\eta + f\nabla_{\rho\alpha}\eta$ = Leibniz rule

This shows that A-connections on A exist.

Also: $R_{\tilde\nabla}(\alpha,\eta) = R_\nabla(\rho\alpha, \rho\eta)$.

(ii) $\bar\nabla_\alpha \eta = \nabla_{\rho\eta}\alpha + [\alpha,\eta]$ ... A-connection on A

$\hookrightarrow \bar\nabla_\alpha f\eta = \underbrace{f\nabla_{\rho\eta}\alpha + f[\alpha,\eta]}_{f\bar\nabla_\alpha\eta} + \rho\alpha(f)\eta$

$\bar\nabla_{f\alpha}\eta = \cancel{(\rho\eta)(f)\alpha} + f\nabla_{\rho\eta}\alpha + f[\alpha,\eta] - \cancel{(\rho\eta)(f)\alpha}$

(iii) $\bar\nabla_\alpha X = \rho(\nabla_X\alpha) + [\rho\alpha, X]$ ... A-connection on TM

$\hookrightarrow \bar\nabla_\alpha fX = f\rho\nabla_X\alpha + f[\rho\alpha, X] + \rho\alpha(f)\cdot X$

$\bar\nabla_{f\alpha}X = f\rho\nabla_X\alpha + \rho(\cancel{(Xf)}\alpha) + f[\rho\alpha, X] - \cancel{(Xf)}\rho\alpha$

The last two are "ρ-compatible":

$$\rho(\bar\nabla_\alpha\eta) = \bar\nabla_\alpha(\rho\eta)$$

Thm. (Levi-Civita connections)

Let $\langle\cdot,\cdot\rangle$ be a (pseudo-) Riemannian metric on A.
There is a unique torsion-free A-connection on A, compatible with $\langle\cdot,\cdot\rangle$, i.e. $\forall\alpha,\eta,\gamma\in\Gamma^\infty(A)$:

$$(\rho\alpha)\langle\eta,\gamma\rangle = \langle\nabla_\alpha\eta,\gamma\rangle + \langle\alpha,\nabla_\eta\gamma\rangle.$$

Pf. Uniqueness

Compute $\rho\alpha\langle\eta,\gamma\rangle + \rho\eta\langle\gamma,\alpha\rangle - \rho\gamma\langle\alpha,\eta\rangle$

$= 2\langle\nabla_\alpha\eta,\gamma\rangle + \langle\eta,[\gamma,\alpha]\rangle + \langle\gamma,[\alpha,\eta]\rangle - \langle\alpha,[\eta,\gamma]\rangle$

i.e. we get the "Koszul formula":

$$\langle\nabla_\alpha\eta,\gamma\rangle = \frac{1}{2}\Big(\langle\alpha,[\eta,\gamma]\rangle - \langle\eta,[\gamma,\alpha]\rangle - \langle\gamma,[\alpha,\eta]\rangle + \rho\alpha\langle\eta,\gamma\rangle + \rho\eta\langle\gamma,\alpha\rangle - \rho\gamma\langle\alpha,\eta\rangle\Big)$$

Since RHS is indep't of ∇, uniqueness follows:

$$\langle \nabla_\alpha \eta - \tilde{\nabla}_\alpha \eta , \gamma \rangle = 0$$

$$\overset{\langle \cdot,\cdot \rangle}{\underset{\text{nondeg.}}{\Longrightarrow}} \quad \nabla_\alpha \eta = \tilde{\nabla}_\alpha \eta \quad \forall \alpha, \eta \in \Gamma^\infty(A).$$

Existence: Define $\nabla_\alpha \eta$ using the Koszul formula and check that ∇ is torsion-free and metric-compatible. $\blacksquare$