Lie groupoids: Lecture 7

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Remark to previous time:

If $\pi : P \xrightarrow{G} M$ and $\tilde{\pi} : \tilde{P} \xrightarrow{\tilde{G}} M$ are two principal bundles and $F : P \to \tilde{P}$ is a morphism (over id_M, with $\varphi : G \to \tilde{G}$) it induces a morphism of Atiyah sequences, i.e.:

$$\begin{tikzcd} 0 \arrow[r] & \frac{P \times \mathfrak{g}}{G} \arrow[r, "\tau"] \arrow[d, "\Theta"'] & \frac{TP}{G} \arrow[r, "\rho"] \arrow[d, "F_*"] & TM \arrow[r] \arrow[d, "id_{TM}"] & 0 \\ 0 \arrow[r] & \frac{\tilde{P} \times \tilde{\mathfrak{g}}}{\tilde{G}} \arrow[r, "\tilde{\tau}"] & \frac{T\tilde{P}}{\tilde{G}} \arrow[r, "\tilde{\rho}"] & TM \arrow[r] & 0 \end{tikzcd}$$

- $F_* : \frac{TP}{G} \to \frac{T\tilde{P}}{\tilde{G}}$, $\quad F_*[v|_u] = [dF_u(v|_u)]$

$$\begin{tikzcd} \frac{TP}{G} \arrow[r] \arrow[d] & \frac{T\tilde{P}}{\tilde{G}} \arrow[d] \\ M \arrow[r, "id"] & M \end{tikzcd}$$

is a morphism of vec. bundles. Right square:

$\tilde{\rho} F_* [v|_u] = \tilde{\rho}[dF_u(v)] = d\tilde{\pi}_{F(u)}(dF_u(v)) = d(\tilde{\pi} \circ F)_u(v)$

$= dF_u(v) = \rho[v|_u]$.

- $\Theta[u, X] := [F(u), \varphi_* X]$. Left square:

$\tilde{\tau}\Theta[u, X] = \tilde{\tau}[F(u), \varphi_* X] = \left[\frac{d}{d\lambda}\Big|_{\lambda=0} F(u) \cdot \underbrace{\exp \lambda \varphi_* X}_{\varphi(\exp \lambda X)}\right]$

$= \left[\frac{d}{d\lambda}\Big|_{\lambda=0} F(u \cdot \exp(\lambda X))\right] = \left[dF\left(\frac{d}{d\lambda}\Big|_{\lambda=0} u \cdot \exp(\lambda X)\right)\right]$

$= F_* \tau[u, X]$.

Finally, F_* preserves the bracket, i.e.

$$F_*[X,Y] = [F_*X, F_*Y] \quad \forall X,Y \in \Gamma^\infty\left(\frac{TP}{G}\right)$$

since $\overline{[X,Y]} = [\bar{X}, \bar{Y}] \in \mathfrak{X}^G(P)$ is F-related to $[\overline{F_*X}, \overline{F_*Y}] = \overline{[F_*X, F_*Y]} \in \mathfrak{X}^{\tilde{G}}(\tilde{P})$.

II.2 The exponential map

Motivation: If G is a Lie gp., then the exponential map is $\exp : \mathfrak{g} \to G$, $\exp(X) = \phi_1^{X^L}(e)$ where $\phi_t^{X^L}$ denotes the flow of left-inv. vector field X^L on G.

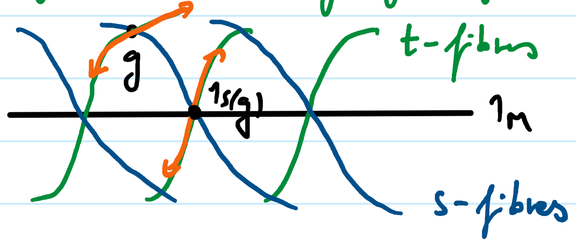
Prerequisite: X^L is complete and there holds $\phi_t^{X^L} = R_{\phi_t^{X^L}(e)}$.

Now let $G \rightrightarrows M$ be a Lie gpd and $X^\alpha \in \mathfrak{X}^L(G)$.

Mimick the trick that shows completeness of a left-inv. vector field on a Lie gp.:

maximal interval

- If $\gamma_g^{X^\alpha} : J_g^{X^\alpha} \to G$ is the integral path of $X^\alpha \in \mathfrak{X}^L(G)$, so is $L_h \circ \gamma_g^{X^\alpha}$, for any $h \in G_{t(g)}$. This implies $J_g^{X^\alpha} \subset J_{hg}$. Picking first $h = g^{-1}$ and secondly $g = 1_{s(g)}$, $h = g$ we get $J_g^{X^\alpha} = J_{1_{s(g)}}^{X^\alpha}$.



- For any $s \in J_{1_x}^{X^\alpha}$, $\xi_s : J_{1_x}^{X^\alpha} - s \to G$ given as $\xi_s(t) = \gamma_{1_x}^{X^\alpha}(t+s)$ is an integral path of X^α, so by maximality of $J_{\xi_s(0)}^{X^\alpha}$ we get $J_{1_x}^{X^\alpha} - s \subset J_{\xi_s(0)}$, but

$\gamma(0)$ doesn't have source equal to x, hence we can't conclude $J^{X^\alpha}_{1_x} - s = J_{1_x}$!

left-inv. fields aren't complete anymore!

<u>Def.</u> A section $\alpha \in \Gamma^\infty(A)$ of a Lie algebroid $A \to M$ is <u>complete</u>, if the vector field $\rho(\alpha) \in \mathfrak{X}(M)$ is complete. $\Gamma^\infty_{cpl}(A) := \{\alpha \in \Gamma^\infty(A) ; \alpha \text{ is complete}\}$.

<u>Lemma.</u> Let $A = A(G)$ for some Lie gpd. $G \rightrightarrows M$, let $\alpha \in \Gamma^\infty(A)$ and $X^\alpha \in \mathfrak{X}^L(G)$ its left-inv. extension. Then α is complete iff X^α is complete.

<u>Prf.</u> • If $A = A(G)$ for some Lie gpd $G \rightrightarrows M$, then completeness of $X^\alpha \in \mathfrak{X}^L(G)$ implies completeness of $\alpha \in \Gamma^\infty(A)$. Indeed, X^α and $\rho\alpha = ds \circ \alpha$ are s-related, which implies (by naturality of flows):

$$\begin{array}{ccc} G & \xrightarrow{s} & M \\ {\scriptstyle \phi^{X^\alpha}_t}\downarrow & & \downarrow{\scriptstyle \phi^{\rho\alpha}_t} \\ G & \xrightarrow[s]{} & M \end{array}$$

i.e. $\phi^{\rho\alpha}_t \circ s = s \circ \phi^{X^\alpha}_t$.

Since s is a submersion, locally $\phi^{\rho\alpha}_t = s \circ \phi^{X_\alpha}_t \circ \sigma$ for any section σ of $s : G \to M$.

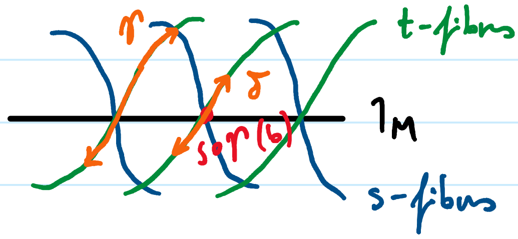
• Conversely, let $\rho\alpha$ be complete and let $\gamma : (a,b) \to G$ be an integral path of X^α; $s \circ \gamma$ is an integral path of $\rho\alpha$, which is complete. If $b < \infty$ we can define

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t) & ; t \in (a,b) \\ \gamma(b-\frac{\varepsilon}{2})\, \delta(b-\frac{\varepsilon}{2})^{-1} \delta(t) & ; t \in (b-\varepsilon, b+\varepsilon) \end{cases}$$

where $\delta : (b-\varepsilon, b+\varepsilon) \to G$ is the integral path of X^α with $\delta(b) = 1_{s\circ\gamma(b)}$. Notice:

Well-def. by completeness of $\rho\alpha$.

- X^α is t-tangent, so δ must lie in $G^{s(\gamma(b))}$



Both $s\circ\gamma$ and $s\circ\delta$ are integral paths of $\rho\alpha$ valued $(s\circ\gamma)(b)$ at b, so they coincide (on whole $\mathbb{R}$).

In particular, $s(\gamma(b-\frac{\varepsilon}{2})) = s(\delta(b-\frac{\varepsilon}{2}))$, so multiplication in def. of $\tilde{\gamma}$ makes sense.

Well-def. of $\tilde{\gamma}$

- $\tilde{\gamma}$ is an integral path of X^α: $\forall t \in (b-\varepsilon, b+\varepsilon)$,

$$\tilde{\gamma}'(t) = d\left(L_{\gamma(b-\varepsilon)\delta(b-\varepsilon)^{-1}}\right)_{\delta(t)}(\delta'(t))$$

$$= X^\alpha|_{\gamma(b-\varepsilon)\delta(b-\varepsilon)^{-1}\delta(t)} = X^\alpha|_{\tilde{\gamma}(t)}$$

$\uparrow$ δ is an integral path of X^α

Hence by uniqueness of integral paths, $\tilde{\gamma}$ is an extension of γ.

We may now define exp:

Def. The <u>exponential map</u> of a Lie gpd. $G \rightrightarrows M$ is the map

$$\exp : \Gamma^\infty_{cpt}(A) \to \mathrm{Bis}(G)$$

$$\exp(\alpha)(x) = \phi_1^{X^\alpha}(1_x)$$

Remark. $\exp(\alpha)$ is a "t-parametrized" bisection, i.e.

$$t\circ\exp(\alpha) = id_M \; , \quad \underbrace{s\circ\exp(\alpha)}_{\text{inverse } s\circ\exp(-\alpha)} : M \xrightarrow{\approx} M .$$

since $X^\alpha \in \ker dt$

Recall from lecture 4 that this implies that the map

$$R_{\exp(\alpha)} : G \to G$$

$$R_{\exp(\alpha)}(g) = g\exp(\alpha)(s(g))$$

is a global right translation, i.e.

$$\begin{tikzcd} G \arrow[rr, "R_{\exp(\alpha)}"] \arrow[dr, "t"'] & & G \arrow[dl, "t"] \\ & M & \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "R_{\exp(\alpha)}"] \arrow[d, "s"] & G \arrow[d, "s"] \\ M \arrow[r, "s\circ\exp(\alpha)"'] & M \end{tikzcd}$$

and

$$R_{\exp(\alpha)}\big|_{G_x} = R_{\exp(\alpha)(x)}$$

and that we also have

$$R_{\exp(\alpha)\exp(\beta)} = R_{\exp(\beta)}\circ R_{\exp(\alpha)} \, ,$$

$$R_{\exp(\alpha)}^{-1} = R_{\exp(-\alpha)} .$$

Properties of exp:

(i) $\phi_t^{X^\alpha}(1_x) = \exp(t\alpha)(x)$

In particular, $\exp(0) = u$ and

$$\frac{d}{dt}\Big|_{t=s}\big(\exp(t\alpha)(x)\big) = X^\alpha\big|_{\exp(s\alpha)(x)} = d\big(L_{\exp(s\alpha)(x)}\big)_{1_x}(\alpha_{1_x})$$

so $\frac{d}{dt}\Big|_{t=0}\big(\exp(t\alpha)(x)\big) = \alpha_{1_x}$.

(ii) $\phi_t^{X^\alpha}(g) = R_{\exp(t\alpha)}(g)$

(iii) The map $\mathbb{R} \to \mathrm{Bis}(G)$, $t \mapsto \exp(t\alpha)$ is a homomorphism of grps, i.e.

$\exp((t+s)\alpha) = \exp(t\alpha)\exp(s\alpha)$. Conversely, any <u>oneparametric Lie subgrpd.</u> $\vartheta : M \times \mathbb{R} \to G$, i.e. $\vartheta(x,\cdot)$ is a homomorphism and $\vartheta(\cdot,t)$ is a bisection obeys $\vartheta(x,t) = \exp(t\vartheta'(0))(x)$, where $\vartheta'(0) \in \Gamma^\infty(A)$ is $\vartheta'(0)_{1_x} = \frac{d}{dr}\Big|_{r=0} \vartheta(x,r)$.

(iv) Naturality of exp:

If $(\Phi, \phi = \mathrm{id}_M)$ is a morphism of Lie grpds $G \rightrightarrows M$, $H \rightrightarrows M$, then:

$$\begin{tikzcd} \mathrm{Bis}(G) \arrow[r, "\sigma \mapsto \Phi \circ \sigma"] & \mathrm{Bis}(H) \\ \Gamma^\infty(\mathfrak{g}) \arrow[u, "\exp_G"] \arrow[r, "\Phi_*"] & \Gamma^\infty(\mathfrak{h}) \arrow[u, "\exp_H"'] \end{tikzcd}$$

$$\Phi(\exp_G(\alpha)(x)) = = \exp_H(\Phi_* \alpha)(x).$$

(Recall $\Phi_* : \Gamma^\infty(\mathfrak{g}) \to \Gamma^\infty(\mathfrak{h})$ is $\Phi_* \alpha = d\Phi|_{\mathfrak{g}} \circ \alpha$.)

Pf. (i) Fix $\lambda \in \mathbb{R}$ and define $\gamma : \mathbb{R} \to G$,

$$\gamma(t) := \phi^{X^\alpha}_{t\lambda}(1_x) = \gamma^{X^\alpha}_{1_x}(t\lambda).$$

Show using chain rule!

This is an integral path of λX^α, starting at $1_x \overset{\text{uniqueness}}{\Longrightarrow} \gamma(t) = \phi^{\lambda X^\alpha}_t(1_x)$. Pick $\lambda = 1$.

(ii) We've already seen that $L_g \circ \gamma^{X^\alpha}_{1_x}$ is an integral path of X^α (provided $s(g) = x$), starting at g.

Uniqueness of integral paths implies

$$\phi_t^{X^\alpha}(g) = g\,\gamma_{1_x}^{X^\alpha}(t) \overset{(i)}{=} g\exp(t\alpha)(s(g)) = R_{\exp(t\alpha)}(g).$$

(iii) Recall: $Bis(G)$ has group structure

$$(\sigma\tilde{\sigma})(x) = \sigma(x)\tilde{\sigma}(s(\sigma(x))), \quad \sigma^{-1}(x) = \sigma((s\circ\sigma)^{-1}(x))^{-1}.$$

$$\exp((\lambda+\mu)\alpha)(x) = \phi_{\lambda+\mu}^{X^\alpha}(1_x) = \phi_\lambda^{X^\alpha}(\phi_\mu^{X^\alpha}(1_x))$$

$$\overset{(ii)}{=} R_{\exp(\lambda\alpha)}(R_{\exp(\mu\alpha)}(1_x)) = \exp(\mu\alpha)(x)\cdot\exp(\lambda\alpha)(s(\ldots))$$

$$= (\exp(\mu\alpha)\exp(\lambda\alpha))(x)$$

For the second part, compute

$$\frac{d}{dt}\Big|_{t=s}\vartheta(x,t) = \frac{d}{dr}\Big|_{r=0}\vartheta(x,s+r)$$

$$= \frac{d}{dr}\Big|_{r=0} L_{\vartheta(x,s)}(\vartheta(x,r))$$

$$= d(L_{\vartheta(x,s)})_{1_x}\Big(\underbrace{\frac{d}{dr}\Big|_{r=0}\vartheta(x,r)}_{\vartheta'(0)_{1_x}}\Big)$$

$$= X^{\vartheta'(0)}\Big|_{\vartheta(x,s)}, \quad \forall s\in\mathbb{R}$$

implying $\vartheta(x,\cdot)$ is the integral path of $X^{\vartheta'(0)}$ starting at 1_x. Hence

$$\vartheta(x,t) = \phi_t^{X^{\vartheta'(0)}}(1_x) = \exp(t\vartheta'(0))(x).$$

(iv) Define $\vartheta(x,t) = \Phi(\exp_G(t\alpha)(x))$; this is a oneparametric Lie subgrpd. of G, with

$$\frac{d}{dt}\Big|_{t=0} \vartheta(x,t) = d\Phi_{1_x}\left(\frac{d}{dt}\Big|_{t=0} \exp_G(t\alpha)(x)\right)$$

$= d\Phi_{1_x}(\alpha_{1_x})$, so $\vartheta'(0) = \Phi_* \alpha$.

$\overset{(iii)}{\Longrightarrow} \vartheta(x,t) = \exp(t\Phi_*\alpha)(x)$. Take $t=1$.

Remark. If σ is an s-parametrized bisection, it follows from (iv) that

$$I_\sigma(\exp(\alpha)(x)) = \exp(Ad_\sigma(\alpha))(x)$$

where $I_\sigma(g) = \sigma(t(g))\, g\, \sigma(s(g))^{-1}$ is the inner automorphism defined by σ, and $Ad_\sigma = (I_\sigma)_*$ is the automorphism of the Lie algebroid $A(G)$, given as its pushforward.