Lie groupoids: Lecture 6

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Continuing with examples of Lie algbds.

(iii) <u>Atiyah algebroid</u> $\pi : P \overset{G}{\to} M$ ... principal bundle.

Let's find the Lie algbd of $\Omega = \frac{P \times P}{G} \rightrightarrows M$.

To do so, we first study the SES of vect. bundles:

$$0 \to \underbrace{\ker d\pi}_{=:VP} \hookrightarrow TP \xrightarrow{(\pi_{TP}, d\pi)} \underbrace{\pi^* TM}_{\{(u,v) \,;\, \pi(u) = \pi_{TM}(v)\}} \to 0$$

The action of G on P induces an action on TP, i.e.

$$T_u P \times G \ni (v, g) \overset{(*)}{\longmapsto} d(r_g)_u(v) \qquad r_g : P \to P,\ u \mapsto u \cdot g$$

- $A := TP/G \xrightarrow{\pi_A} M$, $\pi_A[v|_u] = \pi(u)$ is a vector bundle:
  - the action of G on TP is free and proper, hence TP/G is a smooth mfld s.t. $p_A : TP \to A$ is submersion
  - local trivializations?

    ↳ Let U be a chart domain on M, $P_U := \pi^{-1}(U)$ and $\psi : P_U \to U \times G$ a loc. triv. on $\pi : P \overset{G}{\to} M$. It induces a loc. triv. of $\pi_{TP} : TP \to P$,

$$d\psi : \underbrace{T(P_U)}_{\pi_{TP}^{-1}(P_U)} \to TU \oplus TG \cong (U \times G) \times \overbrace{(\mathbb{R}^{\dim M} \oplus \mathfrak{g})}^{W}$$

$$d\psi(v|_u) = (\psi(u), A(u)v) \qquad A : P_U \to GL(W) \text{ smooth}$$

Then

$$\bar{\psi} : \pi_A^{-1}(U) \to U \times W$$

$$\bar{\psi}\,[v|_u] = \Big(\pi(u),\, pr_2\big(d\psi(v|_u)\big)\Big)$$

$$\begin{tikzcd} \pi_{TP}^{-1}(P_U) \arrow[r, "d\psi"] \arrow[d, "\mu_A"'] & (U\times G)\times W \arrow[d, "pr_U \times id_W"] \\ \pi_A^{-1}(U) \arrow[r, "\bar{\psi}"'] & U \times W \end{tikzcd}$$

is a local triv. for $\pi_A : A \to M$, with inverse

$$\bar{\psi}^{-1}(x, v) = \mu_A\big(d\psi^{-1}((x,e), v)\big)$$

- Furthermore it may be proven that the subbundle $VP = \ker d\pi \subset TP$ is trivial; the trivialization is

$$\tau : P \times \mathfrak{g} \to VP, \quad \tau(u, X) = \frac{d}{d\lambda}\Big|_{\lambda=0} u\cdot\exp(\lambda X).$$

(This is a consequence of the fact that the map $\mathfrak{g} \to \mathfrak{X}(P)$, $X \mapsto \tau(\cdot, X)$ is a monomorphism of Lie alg's.)

So we rather consider the SES

$$0 \to P\times\mathfrak{g} \xrightarrow{\tau} TP \xrightarrow{(\pi_{TP},\, d\pi)} \pi^* TM \to 0.$$

Notice that the action of G on $P \times \mathfrak{g}$, given as

$$(u, X)\cdot g := \big(u\cdot g,\, \underbrace{Ad_{g^{-1}}(X)}_{= d(C_{g^{-1}})_e(X)}\big) \qquad C_{g^{-1}}(h) = g^{-1}hg$$

is equivalent (under the identification τ) to the restriction of the action $(*)$ to $VP \subset TP$:

__Lemma.__ $\tau : P\times\mathfrak{g} \to VP$ is an equivariant isomorphism of vector bundles with respect to G-actions

$$(u,X)\cdot g = (u\cdot g, \mathrm{Ad}_{g^{-1}}(X)) \quad \text{and} \quad v|_u \cdot g = d(r_g)_u(v).$$

__Proof.__ What's left to check is equivariance:

$$\tau((u,X)\cdot g) = \tau(u\cdot g, \mathrm{Ad}_{g^{-1}}(X)) = \frac{d}{d\lambda}\Big|_{\lambda=0} \Big(ug\, \underbrace{\exp(\lambda \mathrm{Ad}_{g^{-1}}(X))}_{C_{g^{-1}}(\exp(\lambda X))}\Big) =$$

$$= \frac{d}{d\lambda}\Big|_{\lambda=0} (u\exp(\lambda X) g) = d(r_g)_u(\tau(u,X)) = \tau(u,X)\cdot g. \quad \blacksquare$$

It isn't hard to show that the __adjoint bundle__

$$\mathrm{Ad}(P) := \frac{P\times\mathfrak{g}}{G}, \quad \pi_{\mathrm{Ad}(P)}[u,X] = \pi(u)$$

is a vector bundle over M (use quotient mfld. thm.)

- We now "take the quotient" of above SES by G:

$$\boxed{0 \to \mathrm{Ad}(P) \xrightarrow{\bar{\tau}} \frac{TP}{G} \xrightarrow{\rho} TM \to 0}$$

ATIYAH SEQUENCE

$$\bar{\tau}[u,X] = [\tau(u,X)], \quad \rho[v|_u] = d\pi(v|_u)$$

This is an exact sequence:

↳
$$\begin{tikzcd} P\times\mathfrak{g} \arrow[r, "\tau"] \arrow[d, "\mu_{\mathrm{Ad}(P)}"'] & TP \arrow[d, "\mu_A"] \\ \mathrm{Ad}(P) \arrow[r, "\bar{\tau}"] & \frac{TP}{G} \end{tikzcd}$$

Since τ is a G-equivariant isomorphism of vec. bundles, it maps $\mu_{\mathrm{Ad}(P)}$-fibres to μ_A-fibres, hence $\bar{\tau}$ is a well-defined injection.

↳ Also, ρ is well-defined:

$$d\pi\left(d(r_g)_u(v|_u)\right) = d(\overbrace{\pi \circ r_g}^{\pi})_u(v_u)$$

$$\begin{tikzcd} TP \arrow[d, "p_A"'] \arrow[dr, "d\pi"] & \\ \frac{TP}{G} =: A \arrow[r, "\rho"'] & TM \end{tikzcd}$$

Hence ρ is a smooth map that is surjective on the fibres.

↳ Exactness in the middle:

$$\ker \overline{d\pi} = \left\{ [v|_u] \in T_u P/G \;;\; \begin{matrix} u \in P, \\ d\pi(v|_u) = 0 \end{matrix} \right\}$$

since τ maps onto $\ker d\pi$, $\overline{\tau}$ maps onto $\ker \overline{d\pi}$, so $\ker \overline{d\pi} = \operatorname{im} \overline{\tau}$.

- To construct a lie bracket on $A = \frac{TP}{G}$, note that we have an isomorphism

$$\mathfrak{X}^G(P) \to \Gamma^\infty\left(\frac{TP}{G}\right), \quad V \mapsto \underline{V}$$

where $\underline{V}|_x := [V|_{u_x}]$, $u_x \in \pi^{-1}(x)$ arbitrary

of $C^\infty(M)$-modules. Here,

$$\mathfrak{X}^G(P) := \left\{ V \in \mathfrak{X}(P) \;;\; d(r_g)_u(V|_u) = V|_{u \cdot g} \;\; \forall g \in G \right\}$$

denotes the set of G-invariant vec. fields on P, and

$$fX := (f \circ \pi) V \quad \forall f \in C^\infty(M),\; V \in \mathfrak{X}^G(P).$$

↳ Indeed, the definition is clearly good and:

$$\begin{tikzcd} P \arrow[r, "V"] \arrow[d, "\pi"'] & TP \arrow[d, "p_A"] \\ M \arrow[r, "\underline{V}"'] & \frac{TP}{G} \end{tikzcd} \qquad \Rightarrow \underline{V} \text{ is smooth.}$$

The inverse to above map is $V \mapsto \bar{V}$, where $\bar{V}|_u \in p_A^{-1}(V|_{\pi(u)})$ is arbitrary.

But $\mathfrak{X}^G(P)$ is closed under the Lie bracket on $\mathfrak{X}(P)$, so we can transfer it to $\Gamma^\infty\left(\frac{TP}{G}\right)$:

$$[V, W] = \underline{[\bar{V}, \bar{W}]} \qquad \forall\, V, W \in \Gamma^\infty\left(\frac{TP}{G}\right)$$

- <u>Compatibility with ρ</u>

  $\bar{V}$ is π-related to ρV (similarly, W)

  hence $[\bar{V}, \bar{W}]$ is π-related to $[\rho V, \rho W]$

  but also $\overline{[V, W]} = [\bar{V}, \bar{W}]$ is π-related to $\rho[V, W]$. Altogether,

  $d\pi[\bar{V}, \bar{W}] = [\rho V, \rho W]$ and $d\pi[\bar{V}, \bar{W}] = \rho[V, W]$

  so $[\rho V, \rho W] = \rho[V, W]$.

- Leibniz rule

$$\overline{[V, fW]} = [\bar{V}, \overline{fW}] = [\bar{V}, f\circ\pi\,\bar{W}]$$

$$= \underbrace{\bar{V}(f\circ\pi)}_{\substack{\rho(V)(f)\circ\pi \\ \text{since } \bar{V} \text{ is } \pi\text{-related} \\ \text{to } \rho V}}\cdot\bar{W} + (f\circ\pi)\cdot[\bar{V},\bar{W}]$$

$$\Rightarrow [V, fW] = \rho(V)(f)W + f[V,W].$$

Theorem. If $\pi : P \xrightarrow{G} M$ is a principal bundle, then $A = \frac{TP}{G}$ is a Lie algbd over M with

$$\rho[V|_u] = d\pi(V|_u), \quad [V,W] := \underline{[\bar{V},\bar{W}]}$$

It fits into the Atiyah sequence and it is the Lie algbd of the gauge gpd.

Pf. We only have to prove the second part.

To see that, note

$$dt_{[u_2,u_1]} : \underbrace{T_{[u_2,u_1]}\Omega}_{= \frac{T_{u_2}P \,\oplus\, T_{u_1}P}{T_{(u_2,u_1)}(\mathrm{Orb}_G(u_2,u_1))}} \to T_{\pi(u_2)}M$$

$$\begin{tikzcd} P\times P \arrow[d] \arrow[dr, "\pi\circ pr_1"] & \\ \Omega \arrow[r, "t"'] & M \end{tikzcd}$$

$$T_{[m,n]}\Omega = \{ (w,v) + \underbrace{T_{(m,n)}\mathrm{Orb}_G(m,n)}_{\{(\frac{d}{dt}|_0 n\cdot\exp tX,\, -n-);\ X\in\mathfrak{g}\}}\ ;\ v,w \in T_n P \}$$

Two cosets $[w,v]$ and $[\tilde{w},\tilde{v}]$ are the same iff
$$w-\tilde{w},\ v-\tilde{v} = \frac{d}{dt}\Big|_{t=0} n\cdot\exp tX \text{ for some } X\in\mathfrak{g}.$$

(In particular, w and $\tilde{w}$ must have the same horizontal part so $d\pi(w) = d\pi(\tilde{w}) \Rightarrow dt$ is well-defined.)

So $[w,v] \in \ker dt_{[n,m]} \iff w = \frac{d}{dt}\big|_{t=0} n\cdot\exp tX$ for some $X\in\mathfrak{g}$.

From this it should be clear that

$[v|_n] \longmapsto [0, v|_n]$ is an iso. $\frac{TP}{G} \to A(\Omega)$ of vector bundles. Since the anchor map is given by
$$ds|_{A(\Omega)}[0,v] = d\pi(v),$$
it should be clear it equals ρ.

Bracket $[\cdot,\cdot]$ is left as exercise. $\blacksquare$