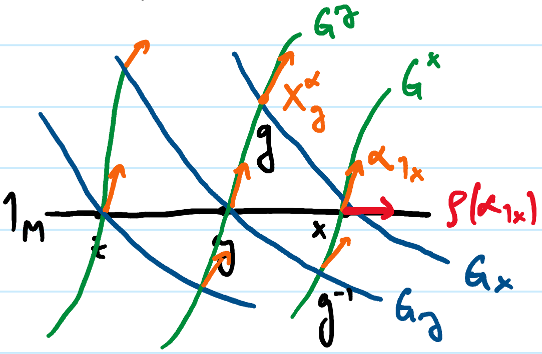
Lie groupoids: Lecture 5

Tuesday, 26 April 2022 13:10

Previous time: the vector space $\mathfrak{X}_L(G)$ of (t-fibre tangential) left-invariant vector fields on G is closed under the Lie bracket and isomorphic (as an $\mathbb{R}$-vector space) to the space $\Gamma^\infty(\mathfrak{g})$ of sections of vector bundle $\mathfrak{g} := \ker(dt)|_{1_M}$ over $1_M \approx M$.

Sketch:



$$X^\alpha_g = d(L_g)_{1_{s(g)}}\left(\alpha|_{1_{s(g)}}\right)$$

We transfer the bracket from $\mathfrak{X}_L(G)$ to $\Gamma^\infty(\mathfrak{g})$ by defining

$$\alpha, \gamma \in \Gamma^\infty(\mathfrak{g}) \rightsquigarrow [\alpha, \gamma]_{1_x} := [X^\alpha, X^\gamma]_{1_x} \quad \forall x \in M.$$

This map $\Gamma^\infty(\mathfrak{g}) \times \Gamma^\infty(\mathfrak{g}) \to \Gamma^\infty(\mathfrak{g})$ obviously satisfies:

- bilinearity
- antisymmetry: $[\alpha, \gamma] = -[\gamma, \alpha]$
- Jacobi identity: $[\alpha, [\gamma, \Upsilon]] + [\gamma, [\Upsilon, \alpha]] + [\Upsilon, [\alpha, \gamma]] = 0$

i.e. it is a <u>Lie bracket</u>. We establish more properties by observing that we also have the <u>anchor map</u>:

$$\rho : \mathfrak{g} \to TM, \quad \rho(\alpha) := ds_{1_x}(\alpha) \quad \forall \alpha \in \mathfrak{g}_{1_x} := \ker dt_{1_x}.$$

i.e. $\rho = ds|_{\mathfrak{g}}$ or:

$$\rho_{1_x} = ds_{1_x} : T_{1_x} G_x \to T_x M$$

Note: we regard $\mathfrak{g}$ as a vector bundle over M since $u : M \to G$ is an embedding; we may think of it as the pullback $u^*\mathfrak{g} := \{(x,\alpha) ;\ \alpha \in \ker dt_{1_x}\}$:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, hook] \arrow[d] & \ker(dt) \arrow[r, "ds"] \arrow[d, "\text{proj.}"] & TM \arrow[d] \\
M \arrow[r, "u"] & G \arrow[r, "s"] & M
\end{tikzcd}$$

The anchor map is a composition of bundle morphisms.

We will also denote $\rho : \Gamma^\infty(\mathfrak{g}) \to \mathfrak{X}(M)$ ... $C^\infty(M)$-linear map of modules

<u>Lemma</u>. The anchor map $\rho : \mathfrak{g} \to TM$ satisfies:

(i) $\rho[\alpha,\gamma]_{\mathfrak{g}} = [\rho\alpha, \rho\gamma]_{TM} \quad \forall \alpha,\gamma \in \Gamma^\infty(\mathfrak{g})$

(ii) The Leibniz identity holds:

$$[\alpha, f\gamma]_{\mathfrak{g}} = f[\alpha,\gamma]_{\mathfrak{g}} + \rho(\alpha)(f)\gamma .$$

$\forall f \in C^\infty(M)$ and $\alpha, \gamma \in \Gamma^\infty(\mathfrak{g})$.

<u>Pf</u>. (i) X^α is s-related to $\rho\alpha$:

$$ds_g(X^\alpha_g) = d(\underbrace{s \circ L_g}_{=s})_{1_{s(g)}}(\alpha_{1_{s(g)}}) = \rho(\alpha_{1_{s(g)}}).$$

and similarly for γ, hence

$[\rho\alpha, \rho\gamma]$ is s-related to $[X^\alpha, X^\gamma]$

i.e. $ds_g[X^\alpha, X^\gamma]_g = [\rho\alpha, \rho\gamma]_{s(g)}$, and

now just take $g = 1_x$.

(iii) $X^{f\alpha}_g = d(L_g)_{1_{s(g)}}(f(s(g))\alpha_{1_{s(g)}}) = (f\circ s)X^\alpha|_g$

so $X^{f\alpha} = (f\circ s)X^\alpha$. Now

$$X^{[\alpha, f\gamma]} = [X^\alpha, X^{f\gamma}] = [X^\alpha, (f\circ s)X^\gamma]$$

$$\overset{\text{usual Leibniz for } [\cdot,\cdot]}{=} (f\circ s)[X^\alpha, X^\gamma] + \underbrace{X^\alpha(f\circ s)}_{ds(X^\alpha)(f)}X^\gamma$$

$$= X^{(f\circ s)[\alpha,\gamma]} + X^{\rho(\alpha)(f)\gamma}$$

$$= X^{(f\circ s)[\alpha,\gamma] + \rho(\alpha)(f)\gamma} \qquad \blacksquare$$

This motivates a general definition:

Def. A <u>Lie algebroid</u> is a vector bundle $A \to M$ together with a vector bundle morphism

$$\begin{tikzcd} A \arrow[rr, "\rho"] \arrow[dr] & & TM \arrow[dl] \\ & M & \end{tikzcd}$$

and a Lie bracket $[\cdot,\cdot]$ on $\Gamma^\infty(A)$ satisfying

- $\rho[\alpha,\gamma] = [\rho\alpha, \rho\gamma]$ $\quad \forall \alpha,\gamma \in \Gamma^\infty(A)$,
- $[\alpha, f\gamma] = f[\alpha,\gamma] + \rho(\alpha)(f)\gamma$ $\quad \forall f \in C^\infty(M)$.

<u>Rmk.</u> How about morphisms? If $(\Phi, \phi = id_M)$ is a morphism

$$\begin{tikzcd} G \arrow[r, "\Phi"] \arrow[d, shift left] \arrow[d, shift right] & H \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, "id_M"] & M \end{tikzcd}$$

of Lie gpds, it induces a morphism $\Phi_* : \mathfrak{g} \to \mathfrak{h}$ of Lie algebroids (i.e. a vector bundle morphism which preserves the brackets & anchors); it is defined on sections as

$$\Gamma^\infty(\mathfrak{g}) \ni \alpha \xmapsto{\Phi_*} d\Phi\big|_{\mathfrak{g}} \circ \alpha \in \Gamma^\infty(\mathfrak{h})$$

which is well-defined (i.e. $\Phi_*\alpha \in \Gamma^\infty(\mathfrak{h})$) since $t^H \circ \Phi = \phi \circ t^G \Rightarrow dt^H(\Phi_*\alpha) = 0$ if α is t^G-vertical. It is a v.b.-morphism since it is $C^\infty(M)$-linear (i.e. $\Phi_*(f\alpha) = f\Phi_*(\alpha)$).

- Lie bracket preservation.

$$\Phi_*\left([\alpha,\beta]_{1_x}\right) = \Phi_*\left([X^\alpha, X^\beta]_{1_x}\right)$$

WTS: $d\Phi|_g [X^\alpha, X^\beta] = [X^{\Phi_*\alpha}, X^{\Phi_*\beta}]$,

so we need to show X^α is Φ-related to $X^{\Phi_*\alpha}$.

$$d\Phi_g\left(X^\alpha|_g\right) = d\Phi_g\left(d(L_g)_{1_x}(\alpha_{1_x})\right)$$

$g: x \to y$

$$= d(\underbrace{\Phi \circ L_g}_{= L_{\Phi(g)} \circ \Phi})_{1_x}(\alpha_{1_x}) = d(L_{\Phi(g)})_{1_x}\left(d\Phi_{1_x}(\alpha_{1_x})\right)$$

$\Phi(1_x) = 1_{\phi(x)} = 1_x$

$$= X^{\Phi_*\alpha}\big|_{\Phi(g)}$$

• From $s^H \circ \Phi = \phi \circ s^G$, we also get the diagram:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\Phi_*"] \arrow[d, "\rho^{\mathfrak{g}}"'] & \mathfrak{h} \arrow[d, "\rho^{\mathfrak{h}}"] \\ TM \arrow[r, "d\phi"'] & TN \end{tikzcd}$$

We will denote $\mathfrak{g}$ or $A(G)$ to denote the algebroid of $G \rightrightarrows M$.

Examples

(0) $TM \to M$ is a Lie algebroid with $\rho = id_{TM}$ and $[\cdot,\cdot] =$ Lie bracket.

Any $\rho : A \to TM$ is a Lie algebroid morphism.

Notice: on any gpd $G \rightrightarrows M$, the map $(t,s) : G \to M \times M$ is a Lie gpd morphism that differentiates to $\rho : A(G) \to TM$;

$$d(t,s)_g : T_g G \to T_{t(g)} M \times T_{s(g)} M,$$
$$v \longmapsto (dt_g(v), ds_g(v))$$

and now $d(t,s)_{1_x}\big|_{\mathfrak{g}} = 0 \oplus \rho$ since $\mathfrak{g} = \ker dt|_{1_M}$

(i) Lie algebras = Lie algebroids with $M = \{*\}$.

(ii) <u>Action Lie algebroids</u>

Let a Lie gp. G act on $M \rightsquigarrow G \ltimes M \rightrightarrows M$.

Then we get a map $a : \mathfrak{g} \to \mathfrak{X}(M)$,

$$a(v)(x) := \frac{d}{dt}\Big|_{t=0} \big(\exp(tv)\cdot x\big)$$

which is a homomorphism of Lie algebras.

The action Lie algbd is the trivial vector bundle

$M \times \mathfrak{g} \to M$ (notice $\Gamma^\infty(M\times\mathfrak{g}) = C^\infty(M,\mathfrak{g})$)

with anchor $\rho(x,v) = a(v)(x)$ and Lie bracket

$\rho : M\times\mathfrak{g} \to TM$

$$[f,g](x) = [f(x),g(x)]_{\mathfrak{g}} + \underbrace{\big(\mathcal{L}_{a(f(x))}\, g\big)(x)}_{dg_x(a(f(x)))} - \big(\mathcal{L}_{a(g(x))}\, f\big)(x)$$

i.e. the Lie bracket on $\Gamma^\infty(M\times\mathfrak{g})$ is uniquely determined by

- $[c_v, c_w] = c_{[v,w]}$; c_v = const. fn into $v\in\mathfrak{g}$
- Leibniz rule.

This is the Lie algebroid of $G \ltimes M = G \times M$:

$$A(G\ltimes M) := (\ker ds)\big|_{\underbrace{1_M}_{M\times\{e\}}} = T_eG \times M = \mathfrak{g}\times M$$

$$\ker ds_{(e,x)} = \{(v,w) \in T_eG \oplus T_xM ;\ ds(v,w) = 0\} = T_eG \oplus 0$$

$s = pr_2$
$\Rightarrow ds(v,w) = w$

- Anchor:

$$dt|_A(v, x) = \frac{d}{d\lambda}\Big|_{\lambda=0} t(\exp(\lambda v), x) = a(v, x)$$

$(v, x) \in \mathfrak{g} \times M$

$$dt|_A\left(\frac{d}{d\lambda}\Big|_{\lambda=0} \exp(\lambda v), x\right)$$

(x in second component denotes the zero section of TM.)

- Lie bracket:

$$[c_v, c_w]_{(e,x)} = [X^{c_v}, X^{c_w}]_{(e,x)} \overset{\text{relatedness}}{=} X^{c_{[v,w]}}_{(e,x)} = c_{[v,w]}|_{(e,x)}.$$

$$X^{c_v}_{(g,x)} := d(R_{(g,x)})_{(e,gx)}(v)$$

So $[c_v, c_w] = c_{[v,w]}$.