Lie groupoids: Lecture 4

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I.3 Bisections

Def. A <u>bisection</u> on $G \rightrightarrows M$ is a smooth map $\sigma : M \to G$ such that $s \circ \sigma = id_M$ and $t \circ \sigma : M \to M$ is a diffeo.

A <u>local bisection</u> on $G \rightrightarrows M$ is a smooth map ($U \overset{open}{\subset} M$) $\sigma : U \to G$ s.t. $s \circ \sigma = id_U$ and $t \circ \sigma : U \to t \circ \sigma(U)$ is a diffeo onto an open subset $t \circ \sigma(U) \subset M$.

We will denote Bis(G) and Bis (G).

Rmk.
- Equivalently, a bisection is an immersed submfld $B \subset G$ such that $s|_B$ and $t|_B$ are diffeomorphisms $B \to M$.
- Bis(G) is a group, with multiplication

  $$(\tilde{\sigma}\sigma)(x) = \underbrace{\tilde{\sigma}(t(\sigma(x)))}_{\text{has source equal to } t(\sigma(x))} \sigma(x)$$

  The unit is u, the inverse of σ is

  $$\sigma^{-1}(x) = \sigma((t \circ \sigma)^{-1}(x))^{-1}$$

- Defining $\bar{\sigma} := \sigma \circ (t \circ \sigma)^{-1}$, we get a smooth map $\bar{\sigma} : M \to G$ with $t \circ \bar{\sigma} = id_M$ and $s \circ \bar{\sigma} =$ diffeo.

  We see $\sigma^{-1} = inv \circ \bar{\sigma}$

Notice: If $\sigma \in$ Bis(G), we get a diffeo.

$$L_\sigma : G \to G, \quad g \mapsto \sigma(t(g))g$$

(Easy to see: $L_\sigma^{-1} = L_{\sigma^{-1}}$, $L_{\tilde{\sigma}\sigma} = L_{\tilde{\sigma}} \circ L_\sigma$)

with the property that $(L_\sigma, t\circ\sigma)$ is a <u>global left translation</u> on G, i.e. diffeo's

$$\begin{tikzcd} G \arrow[rr, "L_\sigma"] \arrow[dr, "s"'] & & G \arrow[dl, "s"] \\ & M & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, "L_\sigma"] \arrow[d, "t"'] & G \arrow[d, "t"] \\ M \arrow[r, "t\circ\sigma"'] & M \end{tikzcd} \quad \text{commute}$$

and $L_\sigma|_{G^x} : G^x \to G^{t\circ\sigma(x)}$ is just $L_{\sigma(x)}$

(the restriction of L_σ to $t^{-1}(x)$ is left trans. $L_{\sigma(x)}$)

Similarly, we get a <u>global right translation</u>

$$R_\sigma(g) := g\,\bar{\sigma}(s(g)),$$

$$\begin{tikzcd} G \arrow[rr, "R_\sigma"] \arrow[dr, "t"'] & & G \arrow[dl, "t"] \\ & M & \end{tikzcd}, \quad \begin{tikzcd} G \arrow[r, "R_\sigma"] \arrow[d, "s"'] & G \arrow[d, "s"] \\ M \arrow[r, "s\circ\bar\sigma"] & M \end{tikzcd}, \quad R_\sigma|_{G_x} = R_{\bar\sigma(x)} \quad (G_x \to G_{s\circ\bar\sigma(x)})$$

with $R_{\tilde\sigma\sigma} = R_\sigma \circ R_{\tilde\sigma}$, $R_\sigma^{-1} = R_{\sigma^{-1}}$.

The <u>inner automorphism</u> defined by σ is $I_\sigma : G \to G$,

$$I_\sigma = L_\sigma \circ R_\sigma^{-1} \quad \left(g \mapsto \sigma(t(g))\, g\, \sigma(s(g))^{-1}\right)$$

$$R_\sigma^{-1}(g) = R_{\sigma^{-1}}(g) = g\,\overline{\sigma^{-1}}(s(g)) \underset{\overline{\sigma^{-1}} = \mathrm{inv}\circ\sigma}{=} g\,\sigma(s(g))^{-1}$$

and notice $(I_\sigma, t\circ\sigma)$ is a Lie gpd. isomorphism:

- $I_\sigma(1_x) = \sigma(x)\sigma(x)^{-1} = 1_{t\circ\sigma(x)}$
- (g,h) composable, i.e. $s(g) = t(h) = x$. Then

$$I_\sigma(gh) = \sigma(\overbrace{t(gh)}^{t(g)})\, gh\, \sigma(\overbrace{s(gh)}^{s(h)})^{-1}$$

squeeze $1_x = \sigma(x)^{-1}\sigma(x)$

Ex. (i) On a trivial gpd $M \times G \times M \rightrightarrows M$, any bisection has the form

$$\sigma(x) = (\varphi(x), \vartheta(x), x)$$

where $\varphi: M \to M$ is a diffeo and $\vartheta \in C^\infty(M, G)$.

(ii) On the action gpd $G \ltimes M$, any bisection has the form $\sigma(x) = (F(x), x)$ where $F: M \to G$ is such that $x \mapsto F(x) \cdot x$ is a diffeo $M \to M$.

(iii) On the gauge gpd $\Omega = \frac{P \times P}{G} \rightrightarrows M$, any bisection σ is a gauge transformation of $\pi: P \xrightarrow{G} M$:

Given $\varphi \in \mathrm{Aut}(P)$, define $\sigma(x) = [\varphi(u), u]$ where $u \in \pi^{-1}(x)$ is arbitrary; this is well-defined and smooth since π is a surjective submersion.

$$\begin{tikzcd} P \arrow[r, "(\varphi, id)"] \arrow[d, "\pi"'] & P \times P \arrow[d] \\ M \arrow[r, "\sigma"] & \Omega \arrow[u, "\rho"', shift right] \end{tikzcd}$$

Notice: $I_\sigma [u_2, u_1] = \sigma(\pi(u_2)) [u_2, u_1] \sigma(\pi(u_1))^{-1}$

$$= [\varphi(u_2), u_2][u_2, u_1][u_1, \varphi(u_1)]$$

$$= [\varphi(u_2), \varphi(u_1)].$$

Conversely if $\sigma \in \mathrm{Bis}(\Omega)$, we fix $x \in M$ and note that $t|_{\Omega_x}: \Omega_x \xrightarrow{\Omega_x^x} M$ is a principal bundle isomorphic to $\pi: P \xrightarrow{G} M$. The map $L_\sigma|_{\Omega_x}: \Omega_x \to \Omega_x$, $[u, u_0] \mapsto \sigma(\pi(u))[u, u_0]$ (fixed $\in \pi^{-1}(x)$) and $\sigma(\pi(u)) = [\varphi(u), u]$ for some uniquely determined $\varphi(u)$ — equivariance is automatic, smoothness follows from the same diagram as above as p and π are surjective submersions. Finally, $f: M \to M$ is determined by $f \circ \pi = \pi \circ \varphi$.

Note: the composition $\Omega_x \xrightarrow{I_\sigma|_{\Omega_x}} \Omega_{t(\sigma(x))} \xrightarrow{R_{\sigma(x)}} \Omega_x$ equals $L_\sigma|_{\Omega_x}$; this is the reason why gauge transformations are also called inner automorphisms.

II. Lie algebroids

We now want to generalize the construction of the Lie algebra of a Lie group G. Recall it is defined as

$$\mathfrak{X}(G) = \{ X \in \mathfrak{X}(G) \,;\, (L_g)_* X = X \ \forall g \in G \}$$

and accompanied w/ the Lie bracket. We have a canonical iso. $\mathfrak{X}(G) \to T_e G =: \mathfrak{g}$, $X \mapsto X_e$ with the inverse $X_e \mapsto (g \overset{X}{\mapsto} d(L_g)_e(X_e))$.

Def. A *left-invariant vector field* on $G \rightrightarrows M$ is $X \in \mathfrak{X}(G)$ with

(i) X is tangent to t-fibres ($X_g \in \ker dt_g \ \forall g \in G$) $X \in \Gamma^\infty(\ker dt)$

(ii) $d(L_g)_h(X_h) = X_{gh} \quad \forall (g,h) \in G * G$.

We denote the set of all ... as $\mathfrak{X}_L(G) \subset \mathfrak{X}(G)$.

Lemma. $\mathfrak{X}_L(G)$ is closed under the Lie bracket.

It is isomorphic to the space of sections $\Gamma^\infty(\mathfrak{g})$ of the vector bundle $\mathfrak{g} := \ker(dt)|_{1_M}$ over 1_M.

Rmk. $\operatorname{rank}(\mathfrak{g}) = \dim G - \dim M$.

emb. submflds

Pf. (i) If X, Y are tangent to t-fibres, so is $[X,Y]$.

For any $g: x \to y$, $L_g : G^x \to G^y$ is a diffeo.

By left-invariance of X,

$(L_g)_*(X|_{G^x}) = X|_{G^y}$ ($X|_{G^x}$ is L_g-related to $X|_{G^y}$)

and similarly, $Y|_{G^x}$ is L_g-related to $Y|_{G^y}$.

This implies $[X,Y]|_{G^x}$ is L_g-related to $[X,Y]|_{G^y}$,

i.e. $d(L_g)_h([X,Y]_h) = [X,Y]_{gh}$ $\forall h \in G^x$.

(ii) The isomorphism is

$\rho : \mathfrak{X}_L(G) \to \Gamma^\infty(\mathfrak{g})$, $\rho(X) = X|_{1_M}$,

its inverse is given by

$\ker(dt)|_{1_M} \ni \alpha \longmapsto X^\alpha$, $X^\alpha_g := d(L_g)_{1_x}(\alpha_{1_x})$ $\quad g: x \to y$

X^α is smooth

We realize X^α as the composition

$$G \xrightarrow{\tau} T(G * G) \xrightarrow{dm} TG$$

$$g \longmapsto (0, \alpha_{1_{s(g)}}) \longmapsto d(L_g)_{1_{s(g)}}(\alpha_{1_{s(g)}})$$

$T_{(g, 1_{s(g)})}(G * G) = \{(v, w) \in T_g G \times T_{1_{s(g)}} G ;\ dt_g(v) = ds_{1_{s(g)}}(w)\}$

(τ is the direct sum of zero section of TG, and α.)

$$(dm \circ \tau)(g) = dm_{(g, 1_{s(g)})}(0, \alpha_{1_{s(g)}})$$

$$= dm_{(g, 1_{s(g)})}(0, \gamma'(0))$$

$\gamma : \mathbb{R} \to G_{s(g)}$

$\gamma(0) = 1_{s(g)}, \ \gamma'(0) = \alpha_{1_{s(g)}}$

$$= \frac{d}{dt}\Big|_{t=0} m(g, \gamma(t)) = \frac{d}{dt}\Big|_{t=0} L_g(\gamma(t))$$

$$= d(L_g)_{1_{s(g)}}(\gamma'(0)) = d(L_g)_{1_{s(g)}}(\alpha_{1_{s(g)}}) \qquad \blacksquare$$