Lie groupoids: Lecture 3

Monday, 4 April 2022 19:37

<u>Proposition</u>. Let $G \rightrightarrows M$ be a Lie grpd.

(i) G_x^y is a closed embedded submfld in G.

(ii) G_x^x is a Lie grp.

(iii) $t: G_x \to \mathrm{Orb}_G(x) := t(G_x)$ is a principal G_x^x-bundle and $\mathrm{Orb}_G(x)$ is an immersed submfld in M.

<u>Proof</u>. (Cont'd)

(ii) We need to check multiplication and inversion are smooth, but that is clear since G_x^x is embedded in G and the images of $m|_{G_x^x \times G_x^x}$ and $\mathrm{inv}|_{G_x^x}$ lie in G_x^x.

$$\begin{tikzcd} G_x^x \times G_x^x \arrow[r, "m"] \arrow[dr, dashed] & G \\ & G_x^x \arrow[u, hook, "\text{embedding}"'] \end{tikzcd} \qquad \begin{tikzcd} G_x^x \arrow[r, "\mathrm{inv}"] \arrow[dr, dashed] & G \\ & G_x^x \arrow[u, hook, "\text{embedding}"'] \end{tikzcd}$$

(!) Restricting the codomain of a smooth map to an immersed submanifold may not yield a smooth fn.

(iii) $\begin{aligned} G_x \times G_x^x &\to G_x \\ (g, h) &\longmapsto gh \end{aligned}$ is a right action.

It's free since if $gh = g$, $h = 1_x$.

The orbits of this action are fibres of $t|_{G_x}: G_x \to \mathrm{Orb}_G(x)$:

If $g: x \to y$, then

$$\mathrm{Orb}_{G_x^x}(g) = \{gh \,;\, h \in G_x^x\} = (t|_{G_x})^{-1}(y) = G_x^y$$

$$\mathrm{Orb}_{G_x^x}(g) = g G_x^x$$

$\supseteq$: $\tilde{g} \in G_x^y \Rightarrow \tilde{g} = g \underbrace{g^{-1}\tilde{g}}_{\in G_x^x}$

Properness: $G * G \to G \times_t G$, $(g,h) \mapsto (g, gh)$ is a diffeo. (established in Lecture 1), hence proper.
Now $G_x \times G_x^x \hookrightarrow G * G$ is proper (since it's a closed embedding), and so is the composition

$$G_x \times G_x^x \hookrightarrow G * G \longrightarrow G \times_t G.$$

Notice that the image of this map lies in $G_x \times_{t|_{G_x}} G_x$, a closed embedded submfld. of $G_x \times G_x$.

This implies G_x / G_x^x is a smooth mfld such that the projection $G_x \to G_x / G_x^x$ is a submersion.

Furthermore,

$$\begin{tikzcd} G_x \arrow[d, "p"'] \arrow[dr, "t|_{G_x}"] & \\ G_x/G_x^x \arrow[r, "\phi"'] & M \end{tikzcd}$$

$\phi(g G_x^x) = t(g)$

$g^{-1}\tilde{g} \in G_x^x \iff t(g) = t(\tilde{g})$

So ϕ is smooth, injective.
Clearly surjective onto $\mathrm{Orb}_G(x)$.

Since p is a submersion and $t|_{G_x}$ has constant rank as shown in (i), ϕ has constant rank as well.
An injective map with constant rank is necessarily an immersion (by rank thm.), so $\mathrm{Orb}_G(x) \subset M$ is an immersed submfld. $\boxtimes$

<u>Cor</u>. For $x \in M$, the division map

$$\delta_x : G_x \times G_x \to G, \quad \delta_x(g,h) = gh^{-1}$$

has const. rank.

Pf. We have the diagram

$$\begin{tikzcd} G_x \times_t G \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & G \arrow[d, "t"] \\ G_x \arrow[r, "t|_{G_x}"'] & M \end{tikzcd}$$

Recall:
$G_x \times_t G =$
$= \{(g,h) \in G_x \times G \,;\, t(g) = t(h)\}$

It's easily verified that:

$$\operatorname{im} d(pr_2)_{(g,h)} = dt_h^{-1}\Big(\overbrace{\operatorname{im} d(t|_{G_x})_g}^{\text{constant dimension by (i)}}\Big)$$

Apply linear-algebraic fact that if $T: V \to W$, $U \leq W$
then $\dim T^{-1}(U) = \dim V + \dim U - \dim(\underbrace{U + \operatorname{im} T}_{= W \text{ if } T \text{ surjective}})$.
Take $T = dt_h$, $U = \operatorname{im} d(t|_{G_x})_g$.

Realize δ_x as the composition

$$G_x \times G_x \xrightarrow{\psi} G_x \times_t G \xrightarrow{pr_2} G$$
$$(g,h) \longmapsto (g, gh^{-1}) \mapsto gh^{-1}$$

and note ψ is a diffeo. (inverse $(g,h) \mapsto (g, h^{-1}g)$). ∎

I.2 Local triviality

Def. A Lie gpd $G \rightrightarrows M$ is transitive, if $\forall x, y \in M \ \exists g \in G_x^y$. Equiv.: $\mathrm{Orb}_G(x) = M \ \forall x \in M$.

It is locally trivial, if $(t,s): G \to M \times M$ is a surjective submersion. "Anchor map"

Rmk. As a (non-Lie) gpd, being transitive forces it to be isomorphic to a trivial gpd: fix $x \in M$ and choose a right inverse $\sigma: M \to G_x$ to $t|_{G_x}$, i.e. $\sigma(y) \in G_x^y$ for any $y \in M$.

The isomorphism is:

$$\begin{tikzcd} M \times G_x^x \times M \arrow[r, "\phi"] \arrow[d, shift left] \arrow[d, shift right] & G \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, "id"] & M \end{tikzcd} \qquad \begin{aligned} &\phi(z, g, y) \\ &= \sigma(z)\, g\, \sigma(y)^{-1} \end{aligned}$$

The appropriate notion of transitivity for Lie gpds is local triviality. Why?

- notice loc. triv. in particular implies transitivity
- gauge gpds are locally trivial by (ii):

Lemma. Let $G \rightrightarrows M$ be a Lie gpd. TFAE:

(i) G is locally trivial.

(ii) $t|_{G_x} : G_x \to M$ is a surjective submersion for one (hence for all) $x \in M$.

(iii) $\delta_x : G_x \times G_x \to G$, $\delta_x(g, h) = g h^{-1}$ is a surjective submersion for one (hence for all) $x \in M$.

Pf. (i→ii) $G_x = (t, s)^{-1}(M \times \{x\})$,

$t|_{G_x}$ = restriction of (t, s) to G_x

(ii→i) $(t, s) \circ \delta_x = t|_{G_x} \times t|_{G_x}$

(iii→ii) $t \circ \delta_x = t|_{G_x} \circ pr_1 : G_x \times G_x \to M$

(ii → iii) Same argument as in corollary, except now $d(t|_{G_x})$ is surjective. $\boxed{\text{/}}$

Ex. On gauge gpd $\pi : P \xrightarrow{G} M$,

$$\begin{tikzcd} P \times \pi^{-1}(x) \arrow[d] \arrow[dr, "pr_1"] & \\ D := \left(\frac{P \times P}{G}\right)_x \arrow[r, "t|_D"'] & M \end{tikzcd}$$

Since pr_1 is a surj. subm., so is $t|_D$.
Local triviality follows.

Rmk. If $G \rightrightarrows M$ is locally trivial, by characterization (ii) we get that there exists an open cover $(U_i)_i$ of M and smooth local sections $\sigma_i : U_i \to G_x$ of $t|_{G_x}$. Thus for any i, we get a map

$$\phi_i : U_i \times G_x^x \times U_i \longrightarrow G_{U_i}^{U_i}$$

$$\phi_i(z, g, y) = \sigma_i(z)\, g\, \sigma_i(y)^{-1},$$

which is an isomorphism of Lie groupoids, i.e. a functor which is a diffeo.

$$\begin{tikzcd} U_i \times G \times U_i \arrow[r, "\phi_i"] \arrow[d, shift left] \arrow[d, shift right] & G_{U_i}^{U_i} \arrow[d, shift left] \arrow[d, shift right] \\ U_i \arrow[r, "id"] & U_i \end{tikzcd}$$

Both on objects and morphisms.

The converse doesn't necessarily hold.

This somewhat justifies the name "local triviality".

Theorem. Let $G \rightrightarrows M$ be a Lie gpd, and $\mathcal{O}(x) := \mathrm{Orb}_G(x)$ for some $x \in M$.

Then $G^{\mathcal{O}(x)}_{\mathcal{O}(x)} \cong$ gauge gpd of $t|_{G_x} : G_x \xrightarrow{G_x^x} \mathcal{O}(x)$ is a Lie subgroupoid of $G \rightrightarrows M$.

In particular, if $G \rightrightarrows M$ is transitive, it's isomorphic to gauge groupoid $t|_{G_x} : G_x \xrightarrow{G_x^x} M$, $\forall x \in M$.

Rmk: A Lie subgpd. of $G \rightrightarrows M$ is a Lie gpd $H \rightrightarrows N$ together with injective immersions $\varphi : N \to M$, $\phi : H \to G$ such that (φ, ϕ) is a functor.

Pf. Define the morphism of Lie gpds as

$$\phi : \frac{G_x \times G_x}{G_x^x} \to G^{\mathcal{O}(x)}_{\mathcal{O}(x)} \subset G \quad \text{and} \quad \varphi : \mathcal{O}(x) \hookrightarrow M \text{ as inclusion.}$$

$$\phi[g,h] = gh^{-1}$$

- well-def.: $\phi[gk, hk] = gkk^{-1}h^{-1} = \phi[g,h]$,
- smooth since $G_x \times G_x \xrightarrow{p} \frac{G_x \times G_x}{G_x^x}$ is submersion,
- injective: $gh^{-1} = \tilde{g}\tilde{h}^{-1} \leadsto \tilde{g} = g \underbrace{h^{-1}\tilde{h}}_{\in G_x^x}$, $\tilde{h} = hh^{-1}\tilde{h}$
- surjective onto $G^{\mathcal{O}(x)}_{\mathcal{O}(x)}$: $k \in G_y^z$ for $y, z \in \mathcal{O}(x)$, so $\exists h : x \to y$ and now $\phi[kh, h] = k$.
- has const. rank since J_x does and p is submersion, thus ϕ is an immersion.

Functoriality:

$$\phi[g_3, g_2]\,\phi[g_2, g_1] = g_3 g_2^{-1} g_2 g_1^{-1} = \phi[g_3, g_1],$$

$$\phi[g, g] = 1_{t(g)}.$$

For the second part, note that if $G \rightrightarrows M$ is transitive, then ϕ is a bijection of const. rank and $\varphi = id_M$. ∎

Cor. A Lie gpd. is transitive iff it's locally trivial.