Lie groupoids: Lecture 2

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Continuing with examples:

iii) <u>Fundamental groupoid</u> $\Pi(M) \rightrightarrows M$ (M connected)

$$\Pi(M) = \{\gamma : [0,1] \to M \text{ piecewise smooth}\} \Big/ \begin{array}{l}\text{relative homotopy}\\ \text{fixing the endpoints}\end{array}$$

- $s[\gamma] = \gamma(0)$, $t[\gamma] = \gamma(1)$

  $1_x = [c_x]$, $[\delta][\gamma] = [\gamma * \delta]$ whenever $\gamma(1) = \delta(0)$

  and $[\gamma]^{-1} = [\bar{\gamma}]$.

- Notice: $\Pi(M)_x \underset{\text{bij.}}{\cong} \tilde{M}$ and $\Pi(M)_x^x = \pi(M, x)$ ← fund. grp.

How to endow $\Pi(M)$ with a topology & smooth structure?

Idea: fix basept. $x \in M$ and let

$$\tilde{M} = \{\gamma : [0,1] \to M \,;\, \gamma(0) = x\} / \sim$$

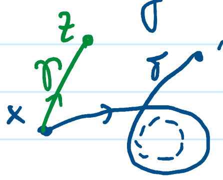
$\gamma \sim \delta \overset{\text{def}}{\iff} \gamma(1) = \delta(1)$ and are relatively homotopic

be the universal cover of M. $\left(q : \tilde{M} \to M,\ q[\gamma] = \gamma(1)\right)$

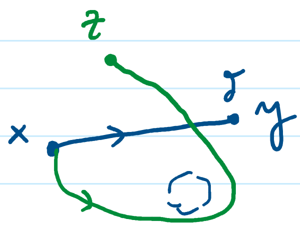
There is a bijection between $\dfrac{\tilde{M} \times \tilde{M}}{Aut(q)}$ and $\Pi(M)$:

$$Aut(q) \cdot ([\gamma], [\delta]) \longmapsto [\bar{\delta} * \gamma].$$

Intuitively:



represents the same class as: $\left(\text{in } \dfrac{\tilde{M} \times \tilde{M}}{Aut(q)}\right)$



So the topology on $\dfrac{\tilde{M} \times \tilde{M}}{Aut(q)}$ can be transferred to $\Pi(M)$. Smoothness: let's consider the following.

iv) <u>Gauge groupoid</u>. Let $\pi : P \xrightarrow{G} M$ be a principal bundle.

(I.e. G acts on P from the right freely, properly, s.t. $\mathrm{orb}_G(u) = \pi^{-1}\pi(u)$ $\forall u \in P$ and π is a surjective submersion) (i.e. G acts on P from the right freely, properly, s.t. $\mathrm{orb}_G(u) = \pi^{-1}\pi(u)$ $\forall u \in P$ and π is a surjective submersion)

G acts on $P \times P$ diagonally: $(u_1, u_2)\cdot g = (u_1 \cdot g, u_2 \cdot g)$.

We get a grpd $\frac{P\times P}{G} \rightrightarrows M$:

- $s[u_2, u_1] = \pi(u_1)$, $t[u_2, u_1] = \pi(u_2)$.
- $1_x = [u, u]$ where $u \in \pi^{-1}(x)$ is arbitrary.
- $[u_3, u_2][u_2, u_1] = [u_3, u_1]$ or more generally:

  $[u_3, u_2'][u_2, u_1] = [u_3, u_1 \delta(u_2', u_2)]$

  defined iff $\pi(u_2') = \pi(u_2)$. Here

  $\delta : P \times_\pi P \to G$ is given by $(u\cdot g, u) \mapsto g$.

  $(\delta(u_2', u_2) = g$, where $u_2' = u_2 \cdot g)$

Intuition:

If $F(E) \xrightarrow{GL(\mathbb{F}, r)} M$ is the frame bundle of a vector bundle $E \to M$ and $u \in F(E)$ is a basis of $E_{\pi(u)}$ (i.e. an isomorphism $\mathbb{F}^r \to E_{\pi(u)}$) then we see $[\tilde{u}, u]$ as the composite iso. $\tilde{u} \circ u^{-1} : E_{\pi(u)} \to E_{\pi(\tilde{u})}$.

Lie grpd structure:

- $\frac{P\times P}{G}$ has a unique smooth structure s.t. $p : P\times P \to \frac{P\times P}{G}$ is a submersion (diagonal action is free & proper) ← since $G \curvearrowright P$ is also such
- s, t are submersions:

$$\begin{tikzcd} P\times P \arrow[d, "p"'] \arrow[dr, "\pi\circ pr_2,\ \pi\circ pr_1"] & \\ \frac{P\times P}{G} \arrow[r, "s,t"'] & M \end{tikzcd}$$

surj. subm. → p

smooth

$\Rightarrow s, t$ are smooth.

Since $\pi \circ pr_1$, $\pi \circ pr_2$ are subm., so are t, s.

- multiplication : abstract argument :

$$
\begin{tikzcd}
P \times (P \times_\pi P) \times P \arrow[rr, "{(u_3, u_2', u_2, u_1) \mapsto (u_3, u_1 \delta(u_2', u_2))}"] \arrow[d, "p \times p"'] \arrow[rrd, bend right, "\text{smooth}"] & & P \times P \arrow[d, "p"] \\
\frac{P \times P}{G} * \frac{P \times P}{G} \arrow[rr, "m"] & & \frac{P \times P}{G}
\end{tikzcd}
$$

surj. subm.

$\Rightarrow$ m is smooth.

Fundamental groupoid :

We realize it as the gauge groupoid of the principal bundle

$$\pi : \tilde{M} \xrightarrow{Aut(q)} M$$

where $Aut(q)$ acts on $\tilde{M}$ from the right as

$$u \cdot \varphi = \varphi^{-1}(u) \qquad (u \in \tilde{M},\ \varphi \in Aut(q)).$$

Concretely : consider $SU(2) \xrightarrow{\mathbb{Z}_2} SO(3)$.

$$\Pi(SO(3)) = \frac{SU(2) \times SU(2)}{\mathbb{Z}_2} = ?$$

Topologically, $\frac{SU(2) \times SU(2)}{\mathbb{Z}_2} \approx SO(4)$; the differ is induced by the map $SU(2) \times SU(2) \to SO(4)$, $(p, q) \mapsto (x \mapsto p x q^{-1})$ where we have identified p, q as unit quaternions and $x \in \mathbb{R}^4$ as a quaternion.

So we get a Lie grpd :

$$SO(4) \rightrightarrows SO(3).$$

Multiplication? Let p, q, p', q' be unit quaternions.

$[p,q][p',q']$ is defined iff q and p' are equal or antipodal (on S^3).

$$\underbrace{[p,q]}_{x \mapsto p x q^{-1}}\underbrace{[q,\tilde{q}]}_{x \mapsto q x \tilde{q}^{-1}} = \underbrace{[p,\tilde{q}]}_{x \mapsto p x \tilde{q}^{-1}}$$

<u>Proposition</u>. Let $G \rightrightarrows M$ be a Lie grpd.

(i) G_x^y is a closed embedded submfld in G.

(ii) G_x^x is a Lie grp.

(iii) $t : G_x \to \mathrm{Orb}_G(x) := t(G_x)$ is a principal G_x^x-bundle.

Recall: let $g : x \to y$. Since $G_x = s^{-1}(x)$, $T_g(G_x) = \ker ds_g$.
Since $G^y = t^{-1}(y)$, $T_g(G^y) = \ker dt_g$.

Also $T_g(G_x), T_g(G^y) \leq T_g(G)$.

$G_x^y = G_x \cap G^y$, but we can't rely on transversality since we don't know if $T_g(G_x) + T_g(G^y) = T_g G$.

<u>Pf</u>. (i) We realize G_x^y as the integral manifolds of the distribution $\ker d(t|_{G_x})$, where $t|_{G_x} : G_x \to M$.

Notice: for any $g : x \to y$, we have

$$\begin{aligned} \ker d(t|_{G_x})_g &= T_g(G_x) \cap \ker dt_g \\ &= T_g(G_x) \cap T_g(G^y) \\ &= \ker ds_g \cap T_g(G^y) = \ker d(s|_{G^y}). \end{aligned}$$

So, fix $x \in M$ and consider $D \subset T(G_x)$, $D_g = \overbrace{\ker(ds_g)}^{= T_g(G_x)} \cap \overbrace{\ker(dt_g)}^{T_g(G^{t(g)})}$.

This is a <u>distribution</u> on $G_{s(g)}$, since if $g : x \to y$:

- $L_g : G^x \to G^y$ is a diffeo, so

$d(L_g)_{1_x} : T_{1_x}(G^x) \to T_g(G^y)$ is an iso.

$d(L_g)_{1_x} : \underbrace{T_{1_x}(G^x)}_{\ker(dt_{1_x})} \to \underbrace{T_g(G^y)}_{\ker(dt_g)}$ is an iso.

- $s|_{G^y} \circ L_g = s|_{G^x}$ on G^x, differentiating at 1_x and taking the kernel:

$$d(L_g)_{1_x}^{-1}\left(\ker d(s|_{G^y})_g\right) = \ker d(s|_{G^x})_{1_x}$$

i.e. $D_g = d(L_g)_{1_x}(D_{1_x})$. This means that $t|_{G_x}$ has constant rank, hence its differential determines a distribution on G_x, which is furthermore trivializable — if $(v_i)_i$ is a basis for D_{1_x}, $X_i(g) := d(L_g)_{1_x}(v_i)$ and $(X_i)_i$ is a global frame for D.

- since D = kernel of differential of smooth map with const. rank, D is involutive (if X, Y are local sections of D on $U \overset{\text{open}}{\subseteq} G_x$, i.e. $d\varphi(X_p) = d\varphi(Y_p) = 0$, then $d\varphi([X,Y]_p) = 0$, so $[X,Y]$ is a local section of D on U) and so by Frobenius' thm. integrable. The leaves of D are connected components of subspaces $\{G_x^y \,;\, y \in M\}$ of G_x, hence G_x^y are immersed submflds. They are embedded because $G_x^y = (t|_{G_x})^{-1}(y)$ is closed in the Hausdorff mfld. G_x, for any $y \in M$.

$\varphi := t|_{G_x}$