Lie groupoids: Lecture 1

Monday, 21 March 2022 13:04

Summary:
- Lie groupoids
- Lie algebroids
- Applications: symplectic geometry, quantum mechanics, ... (noncommutative geometry, index theory, etc.)

Literature:
- Lectures on integrability of Lie brackets (Crainic, Fernandes)
- General theory of Lie gpds and Lie algbds (Mackenzie)

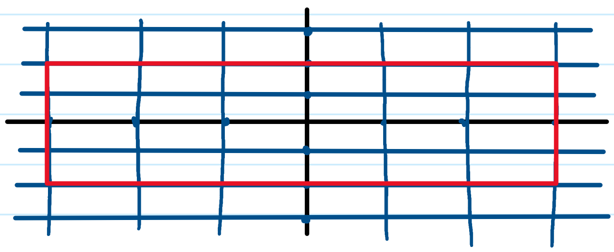
Prerequisites: Smooth manifolds, Diff. geometry, Lie groups

0. Motivation

Let's study symmetries of two spaces:

$$\Omega = (\mathbb{R} \times \mathbb{Z}) \cup (2\mathbb{Z} \times \mathbb{R})$$

$$\tilde{\Omega} = \Omega \cap \big(\underbrace{[-2m, 2m] \times [-n, n]}_{=: A}\big)$$



Symmetry of Ω is captured by action of the group

$$G_\Omega = \{\text{translations by } \Lambda = 2\mathbb{Z} \times \mathbb{Z}\}$$
$$\cup \{\text{reflections through lines in } \tfrac{1}{2}\Lambda\}$$
$$\cup \{\text{reflections through pts in } \tfrac{1}{2}\Lambda\}$$

and likewise, the group

$$G_{\tilde{\Omega}} = \langle a, b \mid a^2 = b^2 = (ab)^2 = e \rangle = D_4 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

is supposed to describe symmetries of $\tilde{\Omega}$.

Notice: G_Ω is large, $G_{\tilde{\Omega}}$ small.

Is this the best we can do to describe $\tilde{\Omega}$?

To study symmetries of $\tilde{\Omega}$, we'd be better off defining:

$$G^{loc}_{\tilde{\Omega}} = \{ (y, \phi, x) \in A \times \underbrace{E(2)}_{O(2) \ltimes \mathbb{R}^2 \text{ (Euclidean gr.)}} \times A \; ; \; y = \phi(x) \text{ and } x \text{ has a nbd. } U \subset \mathbb{R}^2 \text{ s.t.: } \cdot\, \phi(U \cap \tilde{\Omega}) \subset \tilde{\Omega} \;\; \cdot\, \phi(U \cap (A \setminus \tilde{\Omega})) \subset A \setminus \tilde{\Omega} \;\; \cdot\, \phi(U \cap (\mathbb{R}^2 \setminus A)) \subset \mathbb{R}^2 \setminus A \}$$

grid is mapped to grid

open tiles are mapped to open tiles

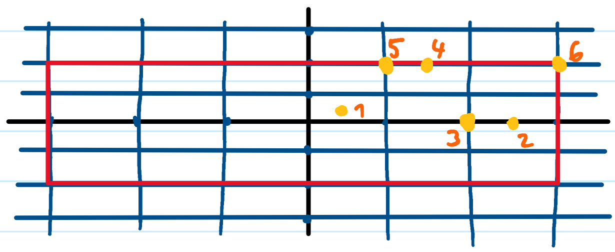
exterior → exterior

This turns out to be a groupoid (moreover, Lie).

Orbits? Naturally we define an orbit as an equivalence class w.r.t. relation $\sim$ on A:

$$x \sim y \overset{\text{def.}}{\Longleftrightarrow} \exists (y, \phi, x) \in G^{loc}_{\tilde{\Omega}}$$

In our case, we get the following orbits:



$\mathcal{O}_1$ = interior pts of tiles 

$\mathcal{O}_2$ = interior edge points 

$\mathcal{O}_3$ = interior crossing points 

$\mathcal{O}_4$ = boundary edge points 

$\mathcal{O}_5$ = boundary "T" points 

$\mathcal{O}_6$ = boundary corner points 

This captures a lot more information than D_4, This captures a lot more information than D_4, since $G^{loc}_{\overline{\Omega}}$ is defined to capture local data.

Another example: actions of $SO(2)$ on $\mathbb{R}^2$ and $SO(3)$ on $\mathbb{R}^3$ $\rightsquigarrow$ $\mathbb{R}^2/SO(2) \approx [0,\infty) \approx \mathbb{R}^3/SO(3)$.

But the singular pt. $\{0\}$ are of different type; this is captured by the fact that the action gpds $SO(2) \circlearrowright \mathbb{R}^2$ and $SO(3) \circlearrowright \mathbb{R}^3$ are not Morita equivalent.

I. Lie groupoids

1. Basic definitions & examples

Df. A groupoid is a small category where every morphism is invertible. More precisely, consists of:

(i) sets G (of morphisms) and M (of objects)

(ii) maps $s, t : G \to M$, prescribing a morphism its domain, codomain

(iii) a unit map $1 : M \to G$ prescribing 1_x to any $x \in M$

(iv) a partial multiplication map "composable pairs"

$$m : G * G \to G \,;\quad G*G = \{(g,h) \in G \times G \,;\, s(g) = t(h)\}$$

which sends (g,h) to gh,

so that

(i) $s(gh) = s(h)$, $t(gh) = t(g)$

(ii) $s(1_x) = t(1_x) = x$, $g 1_{t(g)} = g$, $1_{s(g)} g = g$

(iii) $\forall g \in G \; \exists! \, g^{-1} \in G$. $g^{-1} g = 1_{s(g)}$, $g g^{-1} = 1_{t(g)}$

(iv) $(gh)k = g(hk)$

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Denote: $G \rightrightarrows M$,

G_x^x ... "vertex grp"

$$G_x = s^{-1}(x),\ G^y = t^{-1}(y),\ G_x^y = G_x \cap G^y$$

$$L_g = m(g,\cdot) : G^{s(g)} \to G^{t(g)} \quad (L_g^{-1} = L_{g^{-1}})$$

$$R_g = m(\cdot, g) : G_{t(g)} \to G_{s(g)} \quad (R_g^{-1} = R_{g^{-1}})$$

Prop. $G \rightrightarrows M$, $g \in G_x^y$.

(i) If $hg = g$, then $h = 1_y$. If $gh = g$, then $h = 1_x$

(ii) If $hg = 1_x$ or $gh = 1_y$, then $h = g^{-1}$.

Smooth category?

↳ We want m to be a smooth map, so $G * G$ has to be a smooth manifold. Notice:

$$G * G = (s \times t)^{-1}(\Delta_M) \qquad (s \times t : G \times G \to M \times M)$$

so we require $s \times t$ to be transverse to Δ_M, i.e.

$$d(s \times t)_{(g,h)}(T_g G \oplus T_h G) + T_{(x,x)}\Delta_M = T_{(x,x)}(M \times M)$$

$\forall g, h$. $s(g) = t(h) = x$

or equiv.

$$ds_g(T_g G) \oplus dt_h(T_h G) + \{(v,v)\,;\, v \in T_x M\} = T_x M \oplus T_x M$$

$$\{(v+w, v+u)\,;\, v \in T_x M,\, w \in \operatorname{im}(ds_g),\, u \in \operatorname{im}(dt_h)\} = T_x M \oplus T_x M.$$

This is fulfilled if s,t are submersions.

↳ If only one of them is a submersion, the other one is a submersion if inv is smooth since $s = t \circ \text{inv}$.

Note: $T_{(g,h)}\, G*G = d(s\times t)^{-1}_{(g,h)}\left(T_{(x,x)}\Delta_M\right)$

$$\stackrel{!}{=} \{(v,w) \in T_gG \oplus T_hG \;;\; ds_g(v) = dt_h(w)\}$$

and $\dim G*G = 2\dim G - \dim M$.

Def. A Lie groupoid is a grpd. $G \rightrightarrows M$, s.t. G and M are smooth mflds, $s, t, m, 1_x$ are smooth and s,t are submersions.

Rmk. L_g, R_g are diffeo's but G^x, G_y may not be diffeomorphic since $g: x \to y$ may not exist!

Prop. $\text{inv}: G \circlearrowleft$ in a Lie groupoid is smooth.

Pf. Define

$$\vartheta : G*G \to G\times_t G := (t\times t)^{-1}(\Delta_M)$$

$$\vartheta(g,h) = (g, gh) \quad \text{i.e. } \vartheta = (pr_1, m)$$

and notice this is a bij., since $\vartheta^{-1}(g,h) = (g, g^{-1}h)$.

Realize inv as the composition

$$G \xrightarrow{id\times(1\circ t)} G\times_t G \xrightarrow{\vartheta^{-1}} G*G \xrightarrow{pr_2} G$$

$$g \mapsto (g, 1_{t(g)}) \longmapsto (g, g^{-1}) \mapsto g^{-1}.$$

So WTS ϑ is an immersion (since $\dim G*G = \dim G\times_t G$).

↳ $d\vartheta_{(g,h)}(v,w) = 0$; $pr_1 = pr_1 \circ \vartheta$, so

$\sim G\times_t G$

$\hookrightarrow d\vartheta_{(g,h)}(v,w) = 0$; $pr_1 = pr_1 \circ \vartheta$, so

$$d(pr_1)_{(g,h)}(v,w) = 0.$$

pr_1 ~ $G * G$, pr_1 ~ $G \times_t G$

So $v = 0$. This implies $w \in \ker(dt_h) = T_h G^{t(h)}$

by ! above

But $dm_{(g,h)}(v,w) = d(\underbrace{m(g,\cdot)}_{L_g})_h(w) + d(\underbrace{m(\cdot,h)}_{R_h})_g(v)$

$$= d(L_g)_h(w) \underset{\substack{\uparrow \\ \text{since } m = pr_2 \circ \vartheta}}{=} 0 \xRightarrow{L_g \text{ diffeo}} w = 0.$$

$\boxed{}$

<u>Rmk</u>. Hence we either check s, t are subm. <u>OR</u> one of them is a subm. and inv is smooth.

<u>Examples</u>:

0) <u>Base gpd</u>. $G = \{1_x ; x \in M\}$ $\quad (s,t = id_M)$

i) <u>Trivial gpd</u>; M mfld, G Lie gp. (no action)

Define $M \times G \times M \rightrightarrows M$:

- $s = pr_3$, $t = pr_1$
- $1_x = (x, e, x)$
- $(z, g, y)(y, h, x) = (z, gh, x)$
- $(y, g, x)^{-1} = (x, g^{-1}, y)$.

$M = \{*\}$ ... Lie group ; $G = \{e\}$ ... Pair gpd.

ii) <u>Action gpd</u>. G acts on M from the left. $\underbrace{G \times M \rightrightarrows M}_{G \ltimes M}$:

- $s(g,x) = x$, $t(g,x) = g \cdot x$
- $1_x = (e, x)$
- $(g, y)(h, x) = (gh, x)$ defined iff $y = h \cdot x$

$\cdot\ (g, y)(h, x) = (gh, x)$ defined iff $y = h\cdot x$

$\cdot\ (g, x)^{-1} = (g^{-1}, gx)$

Then: $(G \ltimes M)_x = G \times \{x\}$

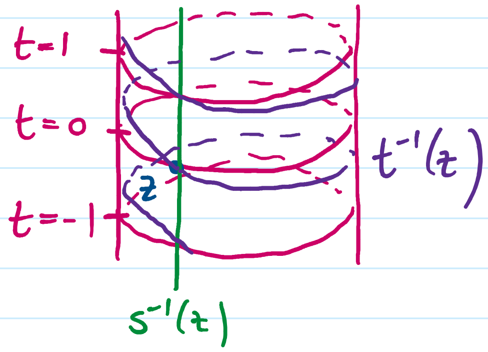
$$(G \ltimes M)^y = \{(g, x) ;\ y = g\cdot x\} =$$

$$= \{(g, g^{-1}y) ;\ g \in G\} \text{... diffeo. to } G$$

Note: $\mathrm{Orb}_G(y) = pr_2((G \ltimes M)^y)$

$$(G \ltimes M)_x^x = \{(g, x) ;\ gx = x\} = \mathrm{Stab}_G(x).$$

Concretely: $\mathbb{R} \ltimes S^1$, $(t, z) \mapsto e^{2\pi i t} z$



$$(\mathbb{R} \ltimes S^1)_z^z = \mathbb{Z} \times \{z\}.$$