Double complexes of representation-valued forms

~ Köln, February 5th 2025 ~

Žan Grad

I. General representations

Motivation : trivial coefficients

- Multiplicative forms on Lie groupoids : $\omega \in \Omega^2(G)$ s.t.

$$\omega(dm(X,Y)) = \omega(X) + \omega(Y) \qquad \forall\, (X,Y) \in T_{(g,h)} G^{(2)}$$

(e.g. multiplicative symplectic structures
... integrated counterparts of Poisson structures)

- Lie groupoid $G \rightrightarrows M$ $\rightsquigarrow$ Bott – Shulman – Stasheff complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
d\uparrow & & d\uparrow & & d\uparrow & & \\
\Omega^1(M) & \xrightarrow{\delta} & \Omega^1(G) & \xrightarrow{\delta} & \Omega^1(G^{(2)}) & \xrightarrow{\delta} & \cdots \\
d\uparrow & & d\uparrow & & d\uparrow & & \\
C^\infty(M) & \xrightarrow{\delta} & C^\infty(G) & \xrightarrow{\delta} & C^\infty(G^{(2)}) & \xrightarrow{\delta} & \cdots
\end{array}$$

Degree q forms on level p of the nerve

$$\boxed{\Omega^q(G^{(p)})}$$

$$\left.\begin{array}{l}\text{Simplicial differential } \delta \\ \text{De Rham differential } d\end{array}\right\} \quad \delta^2 = 0, \quad d^2 = 0, \quad d\delta = \delta d$$

Double complex

Forms with values in representations

- A representation of $G \rightrightarrows M$ is a vector bundle $V \to M$ with a smooth collection of linear iso's $V_{s(g)} \to V_{t(g)}$, $v \mapsto g \cdot v$, $\forall g \in G$
- Representation-valued forms on the nerve:

$$\Omega^{p,2}(G;V) := \Omega^2\left(G^{(p)}; (s \circ pr_p)^* V\right)$$

↳ Still have the canonical simplicial differential:

$$\delta^p : \Omega^{p,2}(G;V) \to \Omega^{p+1,2}(G;V)$$

$$p=0: \quad (\delta^0 \omega)_g = (s^*\omega)_g - g^{-1} \cdot (t^*\omega)_g$$

$$p=1: \quad \delta^1 \omega = pr_2^*\omega - m^*\omega + \phi_* pr_1^*$$

$$\vdots$$

$$p \in \mathbb{N}: \quad \delta^p \omega = \sum_{i=0}^{p} (-1)^i \left(f_i^{(p+1)}\right)^* + (-1)^{p+1} \phi_* \left(f_{p+1}^{(p+1)}\right)^*$$

No canonical de Rham diff.!

ϕ_* changes coefficients, $f_i^{(p+1)} : G^{(p+1)} \to G^{(p)}$ face maps

Infinitesimal counterpart: Weil complex

(Abad, Crainic, Cabrera, Drummond, Meinrenken, Pike, ...)

- $A \Rightarrow M$ Lie algebroid, $\nabla^A : \Gamma(A) \times \Gamma(V) \to \Gamma(V)$ rep. on $V \to M$
- $W^{p,q}(A;V) :=$ sequences $(c_0, \dots, c_p)$, where

$$c_k : \underbrace{\Gamma(A) \times \dots \times \Gamma(A)}_{p-k} \longrightarrow \Omega^{q-k}(M; S^k(A^*) \otimes V)$$

are alternating $\mathbb{R}$-multilinear maps & satisfy

$$c_k(f\alpha_1, \dots, \alpha_{p-k} | \xi_1, \dots, \xi_k) = f c_k(\underline{\alpha} | \underline{\xi}) + df \wedge c_{k+1}(\alpha_2, \dots, \alpha_{p-k} | \alpha_1, \underline{\xi})$$

- Simplicial differential: leading term $(\delta c)_0$ given by Koszul differential

$$(\delta c)_0(\alpha_0, \dots, \alpha_p) = \sum_i (-1)^i \mathcal{L}^A_{\alpha_i}(c_0(\alpha_0, \dots, \hat{\alpha_i}, \dots, \alpha_p))$$
$$+ \sum_{i<j} (-1)^{i+j} c_0([\alpha_i, \alpha_j], \alpha_0, \dots, \hat{\alpha_i}, \dots, \hat{\alpha_j}, \dots, \alpha_p)$$

Here, $\mathcal{L}^A_\alpha$ is induced by the rep. ∇^A. It defines a module structure on $\Omega^m(M; S^n(A^*) \otimes V)$ over the Lie algebra $\Gamma(A)$.

Invariant linear connections (Fernandes, Mărcuț)

- linear connection ∇ on $V \to M$ $\rightsquigarrow$ d^∇ on $\begin{cases} \Omega^{p,q}(G;V) \\ W^{p,q}(A;V) \end{cases}$

  ↳ global : $d^{\nabla^{s\circ pr_p}} \omega$ where $\nabla^f := f^*\nabla$ is the pullback conn.

  ↳ infinitesimal : $(d^\nabla c)_0(\alpha_1, \dots, \alpha_p) = d^\nabla c_0(\alpha_1, \dots, \alpha_p)$

- In general, $\partial d^\nabla \neq d^\nabla \partial$, i.e. d^∇ are not cochain maps.

  <u>Thm.</u> d^∇ is a cochain map on $\Omega^{p,q}(G;V)$ iff

  G-invariance ← $\nabla^s = \nabla^t$ under the identification $s^*V \cong t^*V$.

  <u>Thm.</u> d^∇ is a cochain map on $W^{p,q}(A;V)$ iff

  $$\nabla^A_\alpha = \nabla_{\rho\alpha} \quad \text{and} \quad \iota_{\rho\alpha} R^\nabla = 0. \quad (A\text{-invariance})$$

Fernandes & Mărcuț showed G-invariance $\Rightarrow$ A-invariance

$\Leftarrow$ G s-connected

BSS:

$$\begin{tikzcd}
\Omega^{q+1}(M;V) \arrow[r, "\delta"] & \Omega^{q+1}(G;s^*V) \arrow[r, "\delta"] & \Omega^{q+1}(G^{(2)};(s\circ pr_2)^*V) \arrow[r, "\delta"] & \cdots \\
\Omega^{q}(M;V) \arrow[r, "\delta"'] \arrow[u, "d^\nabla"] & \Omega^{q}(G;s^*V) \arrow[r, "\delta"'] \arrow[u, "d^\nabla"] & \Omega^{q}(G^{(2)};(s\circ pr_2)^*V) \arrow[r, "\delta"'] \arrow[u, "d^\nabla"] & \cdots
\end{tikzcd}$$

Weil:

$$\begin{tikzcd}
\Omega^{q+1}(M;V) \arrow[r, "\delta"] & W^{1,q+1}(A;V) \arrow[r, "\delta"] & W^{2,q+1}(A;V) \arrow[r, "\delta"] & \cdots \\
\Omega^{q}(M;V) \arrow[r, "\delta"'] \arrow[u, "d^\nabla"] & W^{1,q}(A;V) \arrow[r, "\delta"'] \arrow[u, "d^\nabla"] & W^{2,q}(A;V) \arrow[r, "\delta"'] \arrow[u, "d^\nabla"] & \cdots
\end{tikzcd}$$

Feature: $(d^\nabla)^2 \neq 0$ unless ∇ is flat.

Invariance and the van Est map

van Est map: relates global and infinitesimal cochain cpx.

$$VE : \Omega^{p,q}(G;V) \to W^{p,q}(A;V),$$ A = Lie algbd. of G

It satisfies the following properties: (Cabrera, Drummond)

(i) VE is a cochain map: $VE \circ \delta_G = \delta_A \circ VE$.

(ii) The induced map in cohomology is a module homomorphism over $VE : H^\bullet(G) \to H^\bullet(A)$.

(iii) If G is source p_0-connected (homologically), then

$$VE : H^p(\Omega^{\bullet,q}) \xrightarrow{\cong} H^p(W^{\bullet,q}) \quad \forall\, p \le p_0$$

<u>Thm.</u> If ∇ is invariant, $VE \circ d_G^\nabla = d_A^\nabla \circ VE$.

On multiplicative forms, this holds regardless of invariance.

II. Bundles of ideals

Motivation & basic definitions

$$\begin{tikzcd} G \arrow[rr, "\phi"] \arrow[dr, shift left] \arrow[dr, shift right] & & H \arrow[dl, shift left] \arrow[dl, shift right] \\ & M & \end{tikzcd}$$

surjective submersive Lie groupoid morphism ϕ

$\rightsquigarrow$ $\underline{k} := \ker d\phi|_M \subset \ker \rho$ satisfies

$Ad_g(\underline{k}_{s(g)}) = \underline{k}_{t(g)}$ for all $g \in G$.

Infinitesimal realm: $A \Rightarrow M$ Lie algebroid

$\rightsquigarrow$ $\underline{k} \subset \ker \rho$ is a bundle of ideals in A if

$[\alpha, \xi] \in \Gamma(\underline{k})$ for all $\alpha \in \Gamma(A)$, $\xi \in \Gamma(\underline{k})$.

- Regular case: subrepresentations of adjoint repr.

Idea: Instead of using an invariant linear connection we will use a MULTIPLICATIVE EHRESMANN CONNECTION to obtain a richer compatible double cpx. structure.

<u>Global case</u> : fix a bundle of ideals $\underline{k} \subset \ker \rho$ on $G \rightrightarrows M$

• A <u>MEC</u> for $\underline{k}$ is a distribution $E \subset TG$ s.t.

i) E is a subgroupoid of TG

ii) $E \oplus K = TG$ where $K \subset TG$, $K_g = d(L_g)_{1_{s(g)}}(\underline{k}_{s(g)}) = d(R_g)_{1_{t(g)}}(\underline{k}_{t(g)})$

• Equivalent description with 1-forms: $\omega \in \Omega^1(G; s^*\underline{k})$ s.t.

i) ω is multiplicative : $\omega(dm(v,w)) = \omega(w) + Ad_{h^{-1}} \cdot \omega(v)$

ii) $\omega|_{\underline{k}} = id_{\underline{k}}$

Examples:

1) Principal G-bundle $P \to M$.

$$\left\{ \begin{matrix} \text{connections on} \\ P \xrightarrow{G} M \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \text{MEC's for } \underline{k} = \ker \rho \\ \text{on } \mathcal{G}(P) \rightrightarrows M \end{matrix} \right\}$$

2) Vector bundle $E \to M$

$$\left\{ \begin{matrix} \text{connections} \\ \nabla \text{ on } E \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \text{MEC's for } \pi : E \to M \\ \text{on abelian LGB } E \end{matrix} \right\}$$

3) Exists whenever $\underline{k}$ semisimple (fibrewise) or $G \rightrightarrows M$ proper.

Horizontal exterior covariant derivative

↳ A MEC induces:

- Linear connection ∇ on $\underline{k}$, $\nabla_X \xi = v[\underbrace{h}_{TG = E \oplus K}(\underbrace{Y}_{\text{s-lift of } X:\ s_* Y = X}), \xi^L]\big|_M$
- Horizontal projection of forms

  $h^* : \Omega^{p,q}(G; \underline{k}) \to \Omega^{p,q}(G;\underline{k})^{\mathrm{Hor}}$ ← $\iota_X \vartheta = 0$ if $X \in K^{(p)} = K^p \cap TG^{(p)}$

So we may define:

$$D^\omega : \Omega^{p,q}(G;\underline{k}) \to \Omega^{p,q+1}(G;\underline{k}), \qquad D^\omega = h^* \circ d^\nabla$$

<u>Thm.</u> D^ω is a cochain map: $\delta D^\omega = D^\omega \delta$.

In particular, it preserves multiplicativity.

Idea: h^* is a cochain map, so it suffices to show

$$h^*[\delta, d^\nabla] = 0$$

commutator computed explicitly in proof of result regarding invariance

Infinitesimal case: $\underline{k} \subset \ker \rho$ is now a bdl. of ideals in A

- An IM connection is a $\underline{k}$-valued IM 1-form (C, v) s.t.

$$0 \longrightarrow \underline{k} \overset{v}{\longleftarrow}\!\!\longrightarrow A \longrightarrow A/\underline{k} \longrightarrow 0.$$

Here $C : \Gamma(A) \longrightarrow \Omega^1(M; \underline{k})$ contains the data:

i) A connection ∇ on $\underline{k}$: $\nabla \xi = C(\xi)$

ii) A tensor $U : \Gamma(H) \longrightarrow \Omega^1(M; \underline{k})$, $U(\alpha) = C(\alpha)$

$\uparrow$ $A = H \oplus \underline{k}$

$\underline{k}$ is locally trivial!

$\Uparrow$

- IM conditions on (C, v) mean:

i) $\nabla[\xi, \eta] = [\nabla \xi, \eta] + [\xi, \nabla \eta] \quad \forall \xi, \eta \in \Gamma(\underline{k})$

ii) $i_{\rho\alpha} R^{\nabla} \cdot \xi = [\xi, U(h\alpha)] \quad \alpha \in \Gamma(A)$

iii) ... a cocycle-type condition on U

Not hard: $\nabla^A_\alpha - \nabla_{\rho\alpha} = [v\alpha, \cdot]$, so: ∇ is invariant iff $\underline{k}$ abelian!

Induced double complex structure

- To define $D^{(c,v)} : W^{p,q}(A;\underline{k}) \to W^{p,q+1}(A;\underline{k})$, we need

  $h^* : W^{p,q}(A;\underline{k}) \to W^{p,q}(A;\underline{k})^{Hor}$ ← $c_k(\cdot \mid \xi, \cdot) = 0,\ k \geq 1,\ \xi \in \underline{k}$

↳ At $p = 1$, for $c = (c_0, c_1) \in W^{1,q}$:

$$(h^*c)_0(\alpha) = c_0(\alpha) - c_1\, i_C\alpha \quad \text{and} \quad (h^*c)_1(\eta) = c_1(h\eta)$$

We omit the expressions for higher degrees.

Upshot: h^* is a cochain map, $h^*\delta = \delta h^*$.

__Thm.__ $D^{(c,v)} = h^* \circ d^\nabla$ is a cochain map. In particular, the following map preserves infinitesimal multiplicativity:

$$(D^{(c,v)}c)_0(\alpha) = d^\nabla c_0(\alpha) - (c_0 - d^\nabla c_1)\, i_C\alpha$$

$$(D^{(c,v)}c)_1(\eta) = c_0(h\eta) - d^\nabla c_1(h\eta)$$

If G integrates A and $(c,v) = VE(\omega)$:

$$\begin{tikzcd}
\Omega^q_m(G;\underline{k}) \arrow[r, "D^\omega"] \arrow[d, "VE"'] & \Omega^{q+1}_m(G;\underline{k})^{Hor} \arrow[d, "VE"] \\
\Omega^q_{im}(A;\underline{k}) \arrow[r, "D^{(c,v)}"'] & \Omega^{q+1}_{im}(A;\underline{k})^{Hor}
\end{tikzcd}$$

BSS :

$$\begin{tikzcd}
\Omega^{q+1}(M;V) \arrow[r, "\delta"] & \Omega^{q+1}(G;s^*V) \arrow[r, "\delta"] & \Omega^{q+1}(G^{(2)};(s\circ pr_2)^*V) \arrow[r, "\delta"] & \cdots \\
\Omega^{q}(M;V) \arrow[r, "\delta"'] \arrow[u, "d^\nabla"] & \Omega^{q}(G;s^*V) \arrow[r, "\delta"'] \arrow[u, "D^W"] & \Omega^{q}(G^{(2)};(s\circ pr_2)^*V) \arrow[r, "\delta"'] \arrow[u, "D^W"] & \cdots
\end{tikzcd}$$

Weil :

$$\begin{tikzcd}
\Omega^{q+1}(M;V) \arrow[r, "\delta"] & W^{1,q+1}(A;V) \arrow[r, "\delta"] & W^{2,q+1}(A;V) \arrow[r, "\delta"] & \cdots \\
\Omega^{q}(M;V) \arrow[r, "\delta"'] \arrow[u, "d^\nabla"] & W^{1,q}(A;V) \arrow[r, "\delta"'] \arrow[u, "D^{(c,\nabla)}"] & W^{2,q}(A;V) \arrow[r, "\delta"'] \arrow[u, "D^{(c,\nabla)}"] & \cdots
\end{tikzcd}$$

Feature : $\begin{cases} (D^W)^2 \neq 0 \text{ unless } E \subset TG \text{ is involutive}, \\ (D^{(c,\nabla)})^2 \neq 0 \text{ unless } R^\nabla = 0,\ U = 0. \end{cases}$

III. Applications

1. Obstruction to existence of MEC's (Xu, Stiénon, Laurent-Gengoux)

Construction:

- take any $E \subset TG$ s.t. $E \oplus K = TG$ and consider the associated form $\omega \in \Omega^1(G; \underset{\cong}{\overset{K}{s^*\underline{k}}})$
- $\delta\omega$ is a cocycle $(\delta\circ\delta = 0)$ and it is easy to see it vanishes on $K^{(2)}$, i.e. it is horizontal.
- Define $\mathrm{Obs}_{\mathcal{A}(G;\underline{k})} := [\delta\omega] \in H^{2,1}_{Hor}(G;\underline{k})$

  Since $\omega|_K = \mathrm{id}_K$ (under $K \cong s^*\underline{k}$), indep't of E.
- $\mathrm{Obs}_{\mathcal{A}(G;\underline{k})} = 0 \iff$ A MEC exists.

Work in progress: use this to obtain an alternative proof of:

- "$\exists$ MEC for $\underline{k}$" is a Morita invariant $\leftarrow$ since $H^{p,q}_{Hor}(G;\underline{k})$ is
- Existence of MEC's on proper groupoids $\leftarrow$ Vanishing theorem

Infinitesimally: $\mathrm{Obs}_{\mathcal{A}(A;\underline{k})} \in H^{2,1}_{Hor}(A;\underline{k})$.

2. Curvature and affine deformations (infinitesimal case)

- $\Omega^{(C,\nu)} \in \Omega^2_{im}(A; \underline{k})$, $\Omega^{(C,\nu)} := D^{(C,\nu)}(C,\nu)$
- Explicitly:

$$\Omega^{(C,\nu)}\alpha = \left(R^\nabla \cdot \nu\alpha + d^\nabla U(h\alpha),\, U(h\alpha)\right)$$

- Infinitesimal Bianchi identity: $D^{(C,\nu)}\Omega^{(C,\nu)} = 0$
- Affine deformations of IM connections:

Observe: $\mathcal{A}(A; \underline{k})$ is an affine space over $\Omega^1_{im}(A; \underline{k})^{Hor}$

<u>Thm</u>. With an affine deformation

$$(C,\nu) \to (C,\nu) + \lambda(L,\ell),$$

the curvature changes as

$$\Omega^{(C,\nu)+\lambda(L,\ell)} = \Omega^{(C,\nu)} + \lambda D^{(C,\nu)}(L,\ell) + \lambda^2 \kappa_2(L,\ell)$$

where $\kappa_2 : \Omega^1_{im}(A; \underline{k})^{Hor} \to \Omega^2_{im}(A; \underline{k})^{Hor}$,

$$\kappa_2(L,\ell)(\alpha) = -\left(L \wedge L\alpha,\, L(\ell\alpha)\right).$$

3. Primitive multiplicative connections

- IM connection is <u>primitive</u>, if $[\Omega^{(c,v)}] \in H^{1,2}_{Hor}(A;\underline{k})$ vanishes.

  A <u>curving</u> is a choice $F \in \Omega^2(M;\underline{k})$ s.t. $\partial^\circ F = \Omega^{(c,v)}$.

- In this case, the curvature tensors satisfy

  $$R^\nabla \cdot \xi = [\xi, F] \quad \text{and} \quad U(\alpha) = \iota_{\rho\alpha} F$$

  The Bianchi identity becomes

  $$\partial^\circ G = 0, \quad d^\nabla G = 0 \quad \ldots \quad G := d^\nabla F \text{ is the "3-curvature"}$$

- Bijective correspondence : primitive connection + curving

  $\leftrightarrow$ triples (v, ∇, F) such that :

  i) ∇ preserves $[\cdot,\cdot]_{\underline{k}}$

  ii) $R^\nabla = -\mathrm{ad}\, F$

  iii) $d^\nabla F$ is transversal

  $(\iota_{\rho\alpha} d^\nabla F = 0)$

  iv) $\nabla^A_{h\alpha} = \nabla_{\rho\alpha}$

  v) $F^v = (\rho_B)^* F$

  $\Omega^2(B;\underline{k})$, $B = A/\underline{k}$

Examples

1) Transitive algebroids: if $\underline{k} = \ker \rho$, then:

$$\Omega^1_{im}(A;\underline{k})^{Hor} \overset{\delta}{\cong} \Omega^2(M;\underline{k})$$

$\Rightarrow$ All $\underbrace{\text{IM connections}}$ are uniquely primitive!

$\overset{1:1}{\longleftrightarrow}$ splittings of $0 \to \underline{k} \to A \to TM \to 0$

2) $\underline{k}$ has a semisimple typical fibre

$\Rightarrow$ Same conclusion! But now it stems from

$$H^0(\mathfrak{g},\mathfrak{g}) = H^1(\mathfrak{g},\mathfrak{g}) = 0$$

typical fibre

Chevalley-Eilenberg cohomology with values in adjoint rep.

$$\left\{\begin{array}{c}\text{IM}\\ \text{connections}\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{c}\text{Pairs } (\sigma,\nabla) \text{ where } \sigma: A \to \underline{k} \text{ splitting,}\\ \nabla \text{ is bracket-preserving and}\\ \nabla^A_{h\alpha} = \nabla_{\rho\alpha}\end{array}\right\}$$

Thank you!