Duistermaat's integration, II.

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$P(\mathfrak{g})$ as a Banach Lie group

- Banach mfld $\equiv$ modelled on a Banach space. Differentiable means Fréchet differentiable, smooth means that all Fréchet differentials exist and are continuous.

  Banach Lie group: group & Banach mfld., structure maps are smooth.
- Mantra: using implicit/inverse map theorem still works, we still have ODE existence/uniqueness theorems and Frobenius' theorem. But no more local compactness (unless in finite dim's) !
- Lie algebras of Banach Lie groups are construed precisely as in finite dim's. The exponential map still exists (ODE existence) and is a local diffeo at the identity (inverse map theorem).

(Abstractly, a Banach Lie algebra is a Banach space with a Lie bracket, with the additional assumption of its continuity.)

Def. Given a Lie algebra $\mathfrak{g}$, the Banach space $P(\mathfrak{g})$ has the pointwise product:

$$(\gamma_1\gamma_2)(t) = \gamma_1(t) + B_{\gamma_1}(t)\,\gamma_2(t).$$

A path $\gamma \in P(\mathfrak{g})$ is contractible if there is a smooth (Fréchet) map $[0,1] \to P(\mathfrak{g})$, $\varepsilon \mapsto \gamma_\varepsilon$ with $\gamma_0 = 0$, $\gamma_1 = \gamma$ and

$$\int_0^1 B_{\gamma_\varepsilon}(t)^{-1} \frac{\partial \gamma_\varepsilon(t)}{\partial \varepsilon}\, dt = 0 \quad \forall \varepsilon \in [0,1].$$

The set of all such γ is denoted $P(\mathfrak{g})_0$.

Recall: given $\gamma \in P(\mathfrak{g})$, $B_\gamma : [0,1] \to GL(\mathfrak{g})$ is the solution of $B_\gamma'(t)\,B_\gamma(t)^{-1} = \operatorname{ad}\gamma(t)$ with I.C. $B_\gamma(0) = \mathrm{id}_{\mathfrak{g}}$

i.e. $B_\gamma = D^{-1}(\operatorname{ad}\gamma)$ where $D : P(\operatorname{Ad}\mathfrak{g}) \to P(\operatorname{ad}\mathfrak{g})$.

Proposition: (Recall we're giving $\mathfrak{g}$ any norm and $P(\mathfrak{g})$ the induced sup norm.)

(i) $P(\mathfrak{g})$ is a Banach Lie group.

Identity el.: $\gamma = \underline{0}$; inverses $(\gamma^{-1})(t) = -B_\gamma(t)^{-1}\gamma(t)$.

(ii) The Banach Lie algebra $P(\mathfrak{g})^{alg}$ of $P(\mathfrak{g})$ is identified with $P(\mathfrak{g})$.

$$[X,Y](t) = \frac{d}{dt}\Big[\int_0^t X(s)\,ds, \int_0^t Y(s)\,ds\Big] \qquad X, Y \in P(\mathfrak{g})^{alg}$$

is its Lie bracket.

(iii) $P(\mathfrak{g})_0$ is a connected normal Banach Lie subgroup of $P(\mathfrak{g})$ (inclusion is a smooth homomorphism that induces an injective homo. of Lie algebras) whose Banach Lie algebra is

$$P(\mathfrak{g})_0^{alg} = \Big\{ X \in P(\mathfrak{g})^{alg} \;\Big|\; \int_0^1 X(s)\,ds = 0 \Big\}$$

which is the kernel of the Banach Lie algebra homomorphism

$$\operatorname{av} : P(\mathfrak{g})^{alg} \to \mathfrak{g}, \quad \operatorname{av}(X) = \int_0^1 X(s)\,ds.$$

Remark. The interesting thing will be to show that $P(\mathfrak{g})_0 \subset P(\mathfrak{g})$ is actually embedded. (Later; this is where we need the fact that π_2 of any Lie grp is trivial.)

Pf.

(i) Associativity of product:

First off, $\underline{B_{\gamma_1\gamma_2}(t) \overset{(*)}{=} B_{\gamma_1}(t) \circ B_{\gamma_2}(t)}$. (multiplication in $GL(\mathfrak{g})$)

$\hookrightarrow \frac{d}{dt}\big(B_{\gamma_1}(t) \circ B_{\gamma_2}(t)\big)$

$= \operatorname{ad}(\gamma_1(t)) \circ \underline{B_{\gamma_1}(t) \circ B_{\gamma_2}(t)} + \underbrace{\overset{\in \operatorname{Ad}(\mathfrak{g})}{B_{\gamma_1}(t)} \circ \operatorname{ad}(\gamma_2(t))}_{= \operatorname{ad}(B_{\gamma_1}(t)\gamma_2(t)) \circ B_{\gamma_1}(t)} \circ \underline{B_{\gamma_2}(t)}$

$\phi \circ \operatorname{ad}(X) = \operatorname{ad}(\phi(X)) \circ \phi$ i.e. $\phi[X,\cdot] = [\phi X, \phi(\cdot)]$

$= \operatorname{ad}\big(\underbrace{\gamma_1(t) + B_{\gamma_1}(t)\gamma_2(t)}_{\gamma_1\gamma_2(t)}\big) \circ \big(B_{\gamma_1}(t) \circ B_{\gamma_2}(t)\big)$

Also, $B_{\gamma_1}(0) \circ B_{\gamma_2}(0) = \mathrm{id}_{\mathfrak{g}}$. So $(*)$ follows by uniqueness.

So: $(\gamma_1\gamma_2)\gamma_3(t) = \underbrace{\gamma_1\gamma_2(t)}_{\gamma_1(t) + B_{\gamma_1}(t)\gamma_2(t)} + \underbrace{B_{\gamma_1\gamma_2}(t)}_{B_{\gamma_1}(t)B_{\gamma_2}(t)}\gamma_3(t)$

$= \gamma_1(t) + B_{\gamma_1}(t)\big(\gamma_2(t) + B_{\gamma_2}(t)\gamma_3(t)\big) = \gamma_1(\gamma_2\gamma_3)(t)$

In other words: $P(\mathfrak{g}) \to \operatorname{Ad}(\mathfrak{g})$, $\gamma \mapsto B_\gamma(t)$ is a group homo. ($\forall t \in [0,1]$)

Smoothness: The dependence upon γ in definition of B_γ is linear.

Thus the map $P(\mathfrak{g}) \to C^1([0,1], GL(\mathfrak{g}))$, $\gamma \mapsto B_\gamma$ is analytic which in turn makes the group operations analytic.

(Analytic map from an open subset in Banach space into another Banach space $\equiv$ locally looks like a power series ... $f(x) = \sum_{k=0}^{\infty} f_k(x-a)$.) f_k … homogeneous polynomials of deg. k $\Big\}$ $f_n(x) = F(\overset{n}{x, \dots, x})$ for a multilinear map F b/w Banach

(ii) $P(\mathfrak{g})$ is a vector space, so $P(\mathfrak{g})^{alg} := T_{\underline{0}}(P(\mathfrak{g})) = P(\mathfrak{g})$.

To compute the bracket, we recall how to compute the Lie bracket in the finite dimensional case:

$$[X,Y] = \operatorname{ad}(X)(Y) = d(\operatorname{Ad})_e(X)(Y).$$

$\operatorname{Ad} : G \to GL(\mathfrak{g})$, $\operatorname{Ad}_g = d(C_g)_e$

$= \frac{d}{dt}\big|_{t=0} \operatorname{Ad}_{\gamma(t)} Y$ (γ, δ … paths with initial velocities X, Y)

$= \frac{d}{dt}\big|_{t=0} \frac{d}{ds}\big|_{s=0} \underbrace{\gamma(t)\,\delta(s)\,\gamma(t)^{-1}}_{C_{\gamma(t)}\delta(s)}$

I.e. the bracket can be computed with directional derivatives of the conjugation map.

When the Lie grp. is a vector space, we just have to consider $\gamma(t) = e + tX$, $\delta(t) = e + tY$, so this becomes easy.

A similar reasoning applies to Banach Lie groups.

For $\gamma, \tilde{\gamma} \in P(\mathfrak{g})$ we first compute

$(\underset{"(\gamma.\tilde{\gamma}).\gamma^{-1}}{C_\gamma \tilde{\gamma}})(t) = \gamma.\tilde{\gamma}(t) + \underbrace{B_{\gamma.\tilde{\gamma}}(t)}_{B_\gamma(t)B_{\tilde{\gamma}}(t)}\underbrace{\gamma^{-1}(t)}_{-B_\gamma(t)^{-1}\gamma(t)}$

$\overset{(*)}{=} \gamma(t) + B_\gamma(t)\tilde{\gamma}(t) - B_\gamma(t)B_{\tilde{\gamma}}(t)B_\gamma(t)^{-1}\gamma(t)$

We now take the directional derivatives of $(*)$.

To do so, we plug $\tilde{\gamma} = \varepsilon Y$ for $\varepsilon \in \mathbb{R}$ and $Y \in P(\mathfrak{g})^{alg}$, and take $\frac{\partial}{\partial\varepsilon}\big|_{\varepsilon=0}$ of $(*)$. (this is a path in $P(\mathfrak{g})$ with initial velocity Y, starting at $\underline{0}$)

We get:

$$\underset{(\operatorname{Ad}_\gamma Y)(t)}{\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} C_\gamma(\varepsilon Y)(t)} = B_\gamma(t)Y(t) - B_\gamma(t)\underbrace{\Big(\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} B_{\varepsilon Y}(t)\Big)}_{=?}B_\gamma(t)^{-1}\gamma(t)$$

Lemma. For $Y \in P(\mathfrak{g})^{alg}$ there holds:

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \underbrace{B_{\varepsilon Y}(t)}_{\in \operatorname{Ad}(\mathfrak{g}) \subset GL(\mathfrak{g})\ \forall \varepsilon, t} = \operatorname{ad}\int_0^t Y(s)\,ds \in \operatorname{ad}(\mathfrak{g}) \subset \operatorname{End}(\mathfrak{g})$$

Pf. Definition of $B_{\varepsilon Y}$ reads

$$\frac{\partial}{\partial t} B_{\varepsilon Y}(t) = \operatorname{ad}(\varepsilon Y(t)) \circ B_{\varepsilon Y}(t)$$

Differentiate this w.r.t. ε:

$$\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\Big(\frac{\partial}{\partial t} B_{\varepsilon Y}(t)\Big) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\operatorname{ad}(\varepsilon Y(t)) \circ B_{\varepsilon Y}(t)$$

$= \operatorname{ad}(Y(t))$ since $B_{\underline{0}}(t) = \mathrm{id}_{\mathfrak{g}}$ $\forall t$

Now take the $\operatorname{End}(\mathfrak{g})$-valued integral $\int_0^t \text{–}\, dt$ and use the $\operatorname{End}(\mathfrak{g})$-valued Leibniz integral rule

$$\int_0^t \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \frac{\partial}{\partial t} B_{\varepsilon Y}(t)\, dt = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\big(B_{\varepsilon Y}(t) - \cancel{\mathrm{id}_{\mathfrak{g}}}\big). \quad ■$$

So, $(\operatorname{Ad}_\gamma Y)(t) = B_\gamma(t)Y(t) - B_\gamma(t)\Big(\operatorname{ad}\int_0^t Y(s)\,ds\Big)B_\gamma(t)^{-1}\underline{\gamma(t)}$.

Moreover, take $\gamma = \varepsilon X \in P(\mathfrak{g})^{alg}$ and differentiate w.r.t ε, get

$$\underset{[X,Y](t)}{\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \operatorname{Ad}_{\varepsilon X} Y(t)} = \Big(\int_0^t \operatorname{ad} X(s)\,ds\Big)Y(t) - \Big(\int_0^t \operatorname{ad} Y(s)\,ds\Big)X(t)$$

vector-valued integrals are defined so that $\int A f(t)\,dt = A\int f(t)\,dt$ if $A \in \operatorname{End}(V)$, $f : \mathbb{R} \to V$

The only nontrivial contribution in the second term above comes from differentiating $\frac{\partial}{\partial\varepsilon}(\varepsilon X(t))$

$$= \Big[\int_0^t X(s)\,ds, Y(t)\Big] + \Big[X(t), \int_0^t Y(s)\,ds\Big]$$

$$\overset{\text{bilin.}}{=} \frac{d}{dt}\Big[\int_0^t X(s)\,ds, \int_0^t Y(s)\,ds\Big].$$

(iii) The averaging map $\operatorname{av} : P(\mathfrak{g})^{alg} \to \mathfrak{g}$, $\gamma \mapsto \int_0^1 \gamma(s)\,ds$ is a Banach Lie algebra homomorphism on account of (ii) since

$$\int_0^1 \frac{d}{dt}\Big[\int_0^t X(s)\,ds, \int_0^t Y(s)\,ds\Big]dt = \Big[\int_0^1 X(s)\,ds, \int_0^1 Y(s)\,ds\Big]$$

Hence

$$I := \ker \operatorname{av} = \Big\{X \in P(\mathfrak{g})^{alg} \;\Big|\; \int_0^1 X(s)\,ds = 0\Big\}$$

is an ideal in $P(\mathfrak{g})^{alg}$ – in particular, a Banach Lie subalgebra and as such it determines an involutive distribution

$$D \subset T(P(\mathfrak{g})), \quad D_\gamma := d(L_\gamma)_{\underline{0}}(I)$$

on the Banach Lie group $P(\mathfrak{g})$.

By Frobenius' thm. for Banach manifolds, D is integrable.

Claim: The integral (Banach) mfld. of D through $\underline{0}$ is $P(\mathfrak{g})_0$.

$\hookrightarrow$ It's simple to see (using directional derivatives) that

$$\big(d(L_\gamma)_{\underline{0}} X\big)(t) = B_\gamma(t)X(t).$$

Hence:

$\gamma \in P(\mathfrak{g})_0$ iff: $\exists$ smooth path $\varepsilon \mapsto \gamma_\varepsilon$ in $P(\mathfrak{g})$ from $\underline{0}$ to γ (smooth path in path space) which satisfies $\underbrace{\int_0^1 B_{\gamma_\varepsilon}(s)^{-1}\frac{\partial}{\partial\varepsilon}(\gamma_\varepsilon(s))\,ds = 0}$ $\forall\varepsilon$

$\iff d(L_{\gamma_\varepsilon})_{\underline{0}}^{-1}\frac{\partial\gamma_\varepsilon}{\partial\varepsilon} \in I$

$\iff \frac{\partial\gamma_\varepsilon}{\partial\varepsilon} \in D$ i.e. $\varepsilon \mapsto \gamma_\varepsilon$ is tangent to D at time ε

i.e. which is tangent to D

So $\gamma \in P(\mathfrak{g})_0$ iff $\exists$ a smooth path from $\underline{0}$ to γ which is contained within the leaf of D through $\underline{0}$.

This shows that $P(\mathfrak{g})_0$ is an initial immersed connected Banach submfld and as such, it is a Banach Lie subgroup of $P(\mathfrak{g})$.

It is normal since $A = \ker \operatorname{av}$ is an ideal in $P(\mathfrak{g})^{alg}$.

(Equivalence "normal $\iff$ its Lie alg. is ideal" is proven precisely as in fin. dim. case, i.e. by using the exponential map.) ■

Observe: the averaging map $\operatorname{av} : P(\mathfrak{g})^{alg} \xrightarrow{\operatorname{av}} \mathfrak{g}$ is surjective ($\operatorname{const}_X \mapsto X$ does it) and so it induces an isomorphism of Banach Lie algebras

$$P(\mathfrak{g})^{alg} / P(\mathfrak{g})_0^{alg} \cong \mathfrak{g}.$$

Finishing the proof of Lie's third theorem

To conclude Lie III for Lie algebras, we need to show $P(\mathfrak{g})_0$ is an embedded submanifold, so we can then use the following thm.:

If $H \trianglelefteq G$ is a closed normal embedded Banach Lie subgrp of a Banach Lie grp G, then $\exists$ a structure of a Banach manifold on G/H compatible with the quotient topology, such that $G \to G/H$ is a submersive homomorphism of Banach Lie groups. Moreover, $G \to G/H$ is locally trivial w/ fibre H. On the level of Lie algebras, $(G/H)^{alg} \cong \mathfrak{g}/\mathfrak{h}$.

$\hookrightarrow$ Idea for the local triviality part:

Given $x \in G$, submersion normal form thm. for Banach mflds ensures there is a $U \overset{\text{nbhd}}{\ni} x$ and a section $\sigma : \pi(U) \to U$ of π ($\pi \circ \sigma = \mathrm{id}_{\pi(U)}$) through x, i.e. $\sigma(\pi(x)) = x$. ($\pi(U) \overset{\text{open}}{\subset} G/N$)

$\rightsquigarrow \pi(U) \times N \to \pi^{-1}\pi(U)$, $(y, n) \mapsto \sigma(y)n$ is a diffeo.

$\hookrightarrow$ Ref.: H. Glöckner, K. Neeb (2003): Banach-Lie quotients, Enlargibility, and Universal Complexifications (Theorem II.2)

Before proceeding, we note that a closed subgroup of a Banach Lie grp is not in general an embedded Lie subgroup (local compactness issues).

However: If $\phi : G \to H$ is a smooth homomorphism of Banach Lie groups and $T \subset H$ is an embedded Banach Lie subgroup, so is $\phi^{-1}(T)$.

In particular, $\ker\phi$. (We will not prove this result.)

$\hookrightarrow$ Same ref. as above; lemma II.1.

$P(\mathfrak{g})_0 \subset P(\mathfrak{g})$ is embedded

The proof of Lie II from last time readily generalizes to Banach Lie groups.

Idea: If $\mathfrak{g}$ was integrable, we'd use $\boxed{\text{Lie II}}$ on $\operatorname{av} : P(\mathfrak{g})^{alg} \xrightarrow{\operatorname{av}} \mathfrak{g}$ to realize $P(\mathfrak{g})_0$ as the kernel of the obtained homomorphism.

This is motivated by the SES that we've established:

$$0 \to P(\mathfrak{g})_0^{alg} \hookrightarrow P(\mathfrak{g})^{alg} \xrightarrow{\operatorname{av}} \mathfrak{g} \to 0$$

Dropping the assumption of integrability, let's use the adjoint SES to augment the SES above:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \ker(\operatorname{ad}\circ\operatorname{av}) \arrow[d, hook] & \underline{z} \arrow[d, hook] & \\
0 \arrow[r] & P(\mathfrak{g})_0^{alg} \arrow[r, hook] & P(\mathfrak{g})^{alg} \arrow[r, "\operatorname{av}"] \arrow[d, "\operatorname{ad}\circ\operatorname{av}"'] & \mathfrak{g} \arrow[r] \arrow[d, "\operatorname{ad}"] & 0 \\
 & & \operatorname{ad}(\mathfrak{g}) \arrow[r, equal] \arrow[d] & \operatorname{ad}(\mathfrak{g}) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Begin by identifying the integration of $\operatorname{ad}\circ\operatorname{av}$ with the map $\pi : P(\mathfrak{g}) \to \operatorname{Ad}(\mathfrak{g})$, $\pi(\gamma) = B_\gamma(1)$.

$$\hookrightarrow \underset{d\pi_{\underline{0}}(X)}{\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\pi(\varepsilon X)} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} B_{\varepsilon X}(1) = \operatorname{ad}\underbrace{\int_0^1 X(s)\,ds}_{\operatorname{av}(X)}.$$

(Lemma in proof of previous proposition, (ii))

This is a smooth homomorphism of Banach Lie groups with kernel:

$$P(\mathfrak{g})_1 := \ker\pi = \{\gamma \in P(\mathfrak{g}) \mid \underbrace{B_\gamma(1) = \mathrm{id}_{\mathfrak{g}}}_{\text{i.e. } B_\gamma \text{ is a loop in } \operatorname{Ad}(\mathfrak{g})}\} \subset P(\mathfrak{g})$$

… an embedded Banach Lie subgroup with Banach Lie algebra

$$P(\mathfrak{g})_1^{alg} = \ker d\pi_{\underline{0}}$$
$$= \{X \in P(\mathfrak{g})^{alg} \mid \operatorname{ad}(\operatorname{av} X) = 0\}$$
$$= \{X \in P(\mathfrak{g})^{alg} \mid \operatorname{av} X \in \underline{z}\}$$

In particular, we obtain another SES:

$$0 \to P(\mathfrak{g})_0^{alg} \to P(\mathfrak{g})_1^{alg} \xrightarrow{\operatorname{av}} \underline{z} \to 0 \qquad !$$

This restricted map av is where we will use Lie II.

To do so, we must note the following:

- $P(\mathfrak{g})_0 \subset P(\mathfrak{g})_1^\circ$ (connected; connected component) since $P(\mathfrak{g})_0^{alg} \subset P(\mathfrak{g})_1^{alg}$.
- The map $\pi : P(\mathfrak{g}) \to \operatorname{Ad}(\mathfrak{g})$ is surjective: (connected Lie subgp. of $GL(\mathfrak{g})$)

  $\hookrightarrow$ Given $B \in \operatorname{Ad}(\mathfrak{g})$, take a path $\gamma : [0,1] \to \operatorname{Ad}(\mathfrak{g})$ from $\mathrm{id}_{\mathfrak{g}}$ to B.

  Then $D\gamma : [0,1] \to \operatorname{ad}(\mathfrak{g})$, so there is a $\delta \in P(\mathfrak{g})$ such that $\operatorname{ad}(\delta) = D\gamma$. Now just recall that

  $B_\delta := D^{-1}(\operatorname{ad}\delta) = D^{-1}(D\gamma) = \gamma \rightsquigarrow B = \gamma(1) = B_\delta(1)$.

  It's also a submersion since $d\pi_{\underline{0}} = \operatorname{ad}\circ\operatorname{av}$. (surjective)

  Hence $\operatorname{Ad}(\mathfrak{g}) = P(\mathfrak{g})/P(\mathfrak{g})_1$.
- This implies that $\pi : P(\mathfrak{g}) \to \operatorname{Ad}(\mathfrak{g})$ is locally trivial with typical fibre $P(\mathfrak{g})_1$. Then the LES of homotopy groups gives

  $\pi_1(P(\mathfrak{g})_1) \cong \pi_2(\operatorname{Ad}\mathfrak{g})$

  since $P(\mathfrak{g})$ is contractible – a vector space.
- But π_2 of any fin. dim. Lie group is trivial. (!)

  Hence $P(\mathfrak{g})_1^\circ$ is 1-connected and we can use Lie II to integrate $\operatorname{av} : P(\mathfrak{g})_1^{alg} \to \underline{z}$ (commutative Lie algebra) to a smooth homomorphism

  $\phi : P(\mathfrak{g})_1^\circ \to (\mathbb{R}^{\dim\underline{z}}, +)$.

  We know $\ker\phi \subset P(\mathfrak{g})_1^\circ$ is an embedded Banach Lie subgp.

  Now $(\ker\phi)^{alg} = \ker d\phi_{\underline{0}} = \ker\operatorname{av} = P(\mathfrak{g})_0^{alg}$

  and so $P(\mathfrak{g})_0$ = connected component of $\ker\phi$. ■

As noted, Lie III for Lie algebras now follows.

$P(\mathfrak{g})/P(\mathfrak{g})_0$ = integration of $\mathfrak{g}$.

Remark. About the vanishing of π_2 for Lie groups.

$\hookrightarrow$ General argument: use fibration $\Omega G \to P(G) \to G$ where ΩG is the loop space at e. Contractibility of $P(G)$ implies $\pi_k(G) = \pi_{k-1}(\Omega G)$ when passing to the LES of homotopy groups.

Use Morse theory to show $\pi_1(\Omega G) = 0$.

$\hookrightarrow$ Easier for some classical examples of Lie groups:

$SO(n-1) \to SO(n) \to S^{n-1}$

S^{n-1} is $(n-2)$-connected $\overset{\text{LES}}{\implies} \pi_k(SO(n)) = \pi_k(SO(n-1))$ $\forall k < n-2$

So $\pi_2(SO(n)) = \pi_2(SO(n-1))$ $\forall n \geq 5$

Also, $\mathbb{Z}_2 \to S_3 \times S_3 \to SO(4)$ $\implies \pi_2(SO(4)) = 0$

$\mathbb{Z}_2 \to S_3 \to SO(3)$ $\overset{\text{LES}}{\implies} \pi_2(SO(3)) = 0$

And of course $\pi_2(\underbrace{SO(2)}_{S^1}) = 0$.

- Similarly with

  $SU(n-1) \to SU(n) \to S^{2n-1} \implies \pi_2(SU(n)) = \pi_2(SU(n-1))$ $\forall n \geq 3$

  and $SU(2) \cong S^3 \implies \pi_2(SU(2)) = 0$.

  From this we also easily get $\pi_2(U(n)) = 0$ by using the fibration

  $SU(n) \to U(n) \xrightarrow{\det} \mathbb{C}\setminus\{0\}$ ($\simeq S^1$) (also $SO(n) \to O(n) \to \mathbb{R}\setminus\{0\}$ $\simeq \mathbb{Z}_2$)
- Moreover with symplectic groups, use

  $SP(n-1) \to SP(n) \to S^{4n-1}$ and $SP(2) \cong SO(5)$.
- For general/special linear groups, we note that the Gram-Schmidt algorithm gives us a deformation retraction

  $GL(n,\mathbb{C}) \to U(n)$ and $GL(n,\mathbb{R}) \to O(n)$

  and now use the fibration

  $SL(n,\mathbb{F}) \to GL(n,\mathbb{F}) \xrightarrow{\det} \mathbb{F}\setminus\{0\}$

  to also obtain $\pi_2(SL(n,\mathbb{F})) = 0$.

Remark. The integration technique we've presented can be applied to obtain Lie III for Banach Lie algebras:

Thm. (van Est & Korthagen, 1963)

A (Banach) Lie algebra $\mathfrak{g}$ is integrable iff the image of the period homomorphism (quite an involved construction)

$\partial : \pi_2(\operatorname{Ad}\mathfrak{g}) \to \underline{z}$ is discrete.

See Glöckner & Neeb to see how to use Duistermaat's technique of proving this result.