Duistermaat's integration, I.

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We start with a connected Lie grp G whose Lie algebra is $\mathfrak{g}$.
As a motivation, consider the construction of 1-conn. cover.
We normally construct it with continuous paths modulo continuous homotopies with fixed endpoints.
↳ Instead, let's work with C^1-paths and homotopies, which is equivalent since smooth paths/homotopies can be approximated by continuous ones.

$$\tilde{G} := \underbrace{P(G)}_{\substack{C^1\text{-Paths in } G \\ \text{starting at } e}} \Big/ \underset{\substack{\uparrow \\ C^1\text{-homotopies} \\ \text{with fixed} \\ \text{endpoints}}}{\sim} \;=\; P(G) \Big/ \underbrace{P(G)_0}_{\substack{\text{subgroups of} \\ \text{contractible} \\ \text{loops at } e}}$$

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Topological group w.r.t. C^1-topology and pointwise product

The coarsest topology on $C^1(M,N)$ s.t. $C^1(M,N) \longrightarrow C(M,N) \times C(TM,TN)$, $f \mapsto (f, df)$ is continuous, where factors on RHS have compact-open topology.
(We can also consider $C^1(M,N)$ as a subset of RHS.)

Idea: differentiation of paths in $P(G)$ gives continuous paths in $\mathfrak{g} := T_eG$, which we denote $P(\mathfrak{g})$.
Since G is a Lie group, we expect this to be a bijection. Then, we need to see what $P(G)_0$ is mapped to under this differentiation.
In this way, we'll conclude that
$\tilde{G} = P(\mathfrak{g})/P(\mathfrak{g})_0$ for some appropriate $P(\mathfrak{g})_0$.

$P(\mathfrak{g})$ is Banach space with sup-norm w.r.t. any norm on $\mathfrak{g}$. (C^0-topology) } Topologically this is equivalent to the compact-open top. since $[0,1]$ is compact.

Moreover:
- We will prove Lie II using this quotient.
  (This is necessary for Lie III since the proof will also work for Banach-Lie case, and we will use this when dealing with infinite dimensions.)
- $P(\mathfrak{g})$ will turn out to be a Banach Lie group with a certain multiplication, and $P(\mathfrak{g})_0$ its embedded normal Banach Lie subgroup (this is the hard part).
- From this we will get Lie III.

Let us indeed define the differentiation map:

↳ $D : P(G) \to P(\mathfrak{g})$

$$D\gamma(t) = d(R_{\gamma(t)^{-1}})_{\gamma(t)}(\gamma'(t))$$

$$P(G) = \{\gamma \in C^1([0,1], G) \mid \gamma(0) = e\}$$

$$P(\mathfrak{g}) = C^0([0,1], \mathfrak{g})$$

Lemma. If G is connected, D is a homeomorphism.

Proof. Continuity is straightforward since domain has C^1-topology and codomain has compact-open topology.

Let's construct the inverse:

$$\delta \in P(\mathfrak{g}) \rightsquigarrow F_\delta : [0,1] \times G \to TG$$
$$(s, g) \mapsto d(R_g)_e\, \delta(s)$$

Time-dependent vector field on G.

and let γ_δ be the solution to:

$$\begin{cases} \gamma_\delta'(t) = F_\delta(t, \gamma_\delta(t)) \\ \gamma_\delta(0) = e \end{cases} \Rightarrow \quad \delta = D\gamma_\delta \text{ wherever } \gamma_\delta \text{ is defined}$$

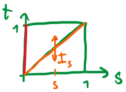
- injectivity of D: clear by uniqueness of ODE solutions
- surjectivity of D: we need to show that γ_δ is defined on $[0,1]$. Denote:

$\Phi_{t,s}$ ... flow of F_δ, i.e. $\frac{d}{dt}\Phi_{t,s}(g) = F_\delta(t, \Phi_{t,s}(g))$

Initial condition: $\Phi_{s,s}(g) = g$

Notice:

- $\gamma_\delta(t) = \Phi_{t,0}(e)$
- Denote



$I = \{ (t,s) \in [0,1]^2 \mid \Phi_{t,s}(e) \text{ is defined} \}$

$I_s = I \cap (\{s\} \times [0,1])$ ... open $\forall s$ by ODE theorem

We want $I_0 = [0,1]$ ... we'll show $I = [0,1]^2$.

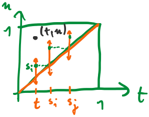
$(t,s),(s,u) \in I$ implies $(t,u) \in I$
$$\left\{ \begin{array}{l} \bullet\ \Phi_{t,s} \circ \Phi_{s,u} = \Phi_{t,u} \quad (\text{time-dep. analogue of } \phi_t \circ \phi_s = \phi_{t+s}) \\ \bullet\ \Phi_{t,s}(g) = \Phi_{t,s}(e)\, g \end{array} \right.$$

$$\hookrightarrow \frac{d}{dt}\left(\Phi_{t,s}(e)_g\right) = d(R_g)\overbrace{\left(\frac{d}{dt}\Phi_{t,s}(e)\right)}^{d(R_{\Phi_{t,s}(e)})_e\, \delta(t)}$$

$$= d(R_{\Phi_{t,s}(e) g})_e\, \delta(t)$$

and $\Phi_{s,s}(e)g = eg = g$.

By compactness of $[0,1]$ there are $0 = s_0 < s_1 < \dots < s_k = 1$ such that $(I_{s_i})_i$ is a cover of $[0,1]$. Hence, $\forall (t,u) \in [0,1]^2$ (WLOG take $t < u$),



we can find a sequence $(t, s_i), (s_i, s_{i+1}), \dots, (s_j, u) \in I$ $\Rightarrow (t,u) \in I$.

- inverse of D is continuous due to continuous dependence of solution of ODE w.r.t vector field (F_δ), and linear dependence upon δ of F_δ. ∎

Let's see what the pointwise product on $P(G)$ translates to with D, also what $P(\mathfrak{g})_0$ should be defined as.

Lemma. G conn. with Lie alg. $\mathfrak{g}$.

(i) $D(\gamma_1\gamma_2)(t) = D\gamma_1(t) + Ad_{\gamma_1(t)} D\gamma_2(t)$ and moreover $Ad_{\gamma(t)} = B_{D\gamma}(t)$

Here we are defining B_δ as the solution $B_\delta : [0,1] \to GL(\mathfrak{g})$ to the initial value problem

Product on $P(\mathfrak{g})$: $\rightsquigarrow (\delta_1\delta_2)(t) := \delta_1(t) + B_{\delta_1}(t)\,\delta_2(t)$

We'll check associativity next time.

$$\begin{cases} B_\gamma'(t)\,B_\gamma(t)^{-1} = \operatorname{ad}\gamma(t) \\ B_\gamma(0) = \mathrm{id}_{\mathfrak{g}} \end{cases}$$

Image of $\operatorname{ad}: \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$
... subalgebra of $\operatorname{End}(\mathfrak{g})$

In other words, $\boxed{B_\gamma := D^{-1}(\operatorname{ad}\gamma)}$ where $D : P(\underbrace{\operatorname{Ad}(\mathfrak{g})}) \to P(\overbrace{\operatorname{ad}(\mathfrak{g})})$.

So actually $B_\gamma : [0,1] \to \operatorname{Ad}(\mathfrak{g})$ maps into the adjoint group; it is the integral path of right-invariant vec. field $(\operatorname{ad}\gamma)^R$.

Subgroup of $GL(\mathfrak{g})$ generated by $e^{\operatorname{ad}X} = \operatorname{Ad}_{\exp(X)}$, $X \in \mathfrak{g}$.
... connected Lie subgrp of $GL(\mathfrak{g})$ which integrates $\operatorname{ad}(\mathfrak{g})$.

(ii) A smooth homotopy $\varepsilon \mapsto \Upsilon_\varepsilon \in P(G)$ has fixed endpoint $(\Upsilon_\varepsilon(1) = \text{const.})$ iff:

$$\boxed{\int_0^1 B_{\gamma_\varepsilon}(t)^{-1}\,\frac{\partial\gamma_\varepsilon(t)}{\partial\varepsilon}\,dt = 0 \quad \forall\varepsilon} \quad (*) \quad \left(\mathfrak{g}\text{-valued integral}\right)$$

where $\gamma_\varepsilon = D\Upsilon_\varepsilon$.

Note: $\dfrac{\partial\gamma_\varepsilon(t)}{\partial\varepsilon} \in T_{\gamma_\varepsilon(t)}\mathfrak{g} \cong \mathfrak{g}$

We're using the canonical identification.

In particular, $D(\overbrace{P(G)}^{\text{contractible loops at } e}{}_0)$ is equal to the subset of $P(\mathfrak{g})$ of all such γ for which $\exists$ a map $[0,1] \to P(\mathfrak{g})$, $\varepsilon \mapsto \gamma_\varepsilon$ satisfying $(*)$ and $\gamma_0 = \gamma$ and $\gamma_1 = 0$.

definition of $P(\mathfrak{g})_0$

Remark.

Instead of the pointwise product on $P(G)$, we can use a concatenation

$$(g_1 \cdot g_2)(t) = \begin{cases} g_2(2t) & 0 \le t \le 1/2 \\ g_1(2t-1)\,g_2(1) & 1/2 \le t \le 1 \end{cases}$$

and we will get a concatenation on $P(\mathfrak{g})$:

$$D(g_1 \cdot g_2)(t) = \begin{cases} 2Dg_2(2t) & 0 \le t \le 1/2 \\ 2Dg_1(2t-1) & 1/2 \le t \le 1 \end{cases}$$

However, we then need to work with paths $P(G)$ which have all derivatives vanishing at endpoints. Moreover, this product is associative only up to homotopy, whereas the one we're using is associative on the nose.

(this is actually what MC & RLF do instead.)

Proof.

(i) $$\frac{d}{dt}\left(\gamma_1(t)\gamma_2(t)\right) = d(R_{\gamma_2(t)})\overbrace{\gamma_1'(t)}^{d(R_{\gamma_1(t)})(D\gamma_1(t))} + d(L_{\gamma_1(t)})\overbrace{\gamma_2'(t)}^{d(R_{\gamma_2(t)})(D\gamma_2(t))}$$

$$= d(R_{\gamma_1\gamma_2})\left(D\gamma_1(t) + \underbrace{d(R_{\gamma_1(t)^{-1}})\,d(L_{\gamma_1(t)})}_{Ad_{\gamma_1(t)}} D\gamma_2(t)\right)$$

This proves the first part.

To show $\underline{Ad_{\gamma(t)} = B_{D\gamma}(t)}$, compute

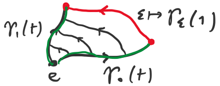
$$\frac{d}{ds}\Big|_{s=t} Ad_{\gamma(s)}\,Ad_{\gamma(t)^{-1}} = \frac{d}{ds}\Big|_{s=t} Ad_{\gamma(s)\gamma(t)^{-1}}$$

$$= ad\left(\frac{d}{ds}\Big|_{s=t}\gamma(s)\gamma(t)^{-1}\right)$$

$$= ad\left(D\gamma(t)\right)$$

and of course (IC): $Ad_{\gamma(0)} = Ad_e = id_{\mathfrak{g}}$.

(ii) We want to consider the path $\varepsilon \mapsto \gamma_\varepsilon(1)$



$$\frac{\partial}{\partial\varepsilon}\gamma_\varepsilon(1) = ?$$

Fix $t, \varepsilon \in [0,1]$ and introduce $F : [0,1]^2 \to G$,

$$F(r,\vartheta) = \gamma_\varepsilon(r)^{-1}\gamma_\vartheta(r)\gamma_\vartheta(t)^{-1}\gamma_\varepsilon(t)$$

If $r = t$ or $\vartheta = \varepsilon$, we just get e:

$$F(t,\vartheta) = F(r,\varepsilon) = e$$

which implies (t,ε) is a critical pt. of F, i.e.

$$\frac{\partial F}{\partial r}(t,\varepsilon) = \frac{\partial F}{\partial \vartheta}(t,\varepsilon) = 0.$$

So we can consider the Hessian of F, which is symmetric, i.e.

$$\frac{\partial}{\partial r}\Big|_{r=t}\frac{\partial}{\partial\vartheta}\Big|_{\vartheta=\varepsilon} F = \frac{\partial}{\partial\vartheta}\Big|_{\vartheta=\varepsilon}\frac{\partial}{\partial r}\Big|_{r=t} F$$

/ In the end, $\int_0^1 \dots dt$

$$\text{LHS} = \frac{\partial}{\partial r}\Big|_{r=t}\Big(\frac{\partial}{\partial\vartheta}\Big|_{\vartheta=\varepsilon} \gamma_\varepsilon(r)^{-1}\gamma_\vartheta(r) + \frac{\partial}{\partial\vartheta}\Big|_{\vartheta=\varepsilon} \underset{\text{no } r}{\cancel{\gamma_\vartheta(t)^{-1}\gamma_\varepsilon(t)}}\Big)$$

$$= \frac{\partial}{\partial r}\Big|_{r=t} d\big(L_{\gamma_\varepsilon(r)^{-1}}\big)\frac{\partial\gamma_\varepsilon(r)}{\partial\varepsilon}$$

$$= \frac{\partial}{\partial t}\Big(d\big(L_{\gamma_\varepsilon(t)^{-1}}\big)\frac{\partial}{\partial\varepsilon}\gamma_\varepsilon(t)\Big)$$

$$\text{RHS} = \frac{\partial}{\partial\vartheta}\Big|_{\vartheta=\varepsilon}\Big(\frac{\partial}{\partial r}\Big|_{r=t} \gamma_\varepsilon(t)^{-1}\gamma_\vartheta(r)\gamma_\vartheta(t)^{-1}\gamma_\varepsilon(t) + \frac{\partial}{\partial r}\Big|_{r=t} \underset{\text{no } \vartheta}{\cancel{\gamma_\varepsilon(r)^{-1}\gamma_\varepsilon(t)}}\Big)$$

$$= d\big(L_{\gamma_\varepsilon(t)^{-1}}\big)\, d\big(R_{\gamma_\varepsilon(t)}\big)\frac{\partial}{\partial\vartheta}\Big|_{\vartheta=\varepsilon}\underbrace{\frac{\partial}{\partial r}\Big|_{r=t}\big(\gamma_\vartheta(r)\gamma_\vartheta(t)^{-1}\big)}_{=D\gamma_\vartheta = \delta_\vartheta}$$

$$= \mathrm{Ad}_{\gamma_\varepsilon(t)^{-1}}\frac{\partial\delta_\varepsilon(t)}{\partial\varepsilon}$$

Integrating $\int_0^1 \dots dt$:

$$\int_0^1 \underbrace{\mathrm{Ad}_{\gamma_\varepsilon(t)^{-1}}}_{B_{\delta_\varepsilon(t)^{-1}} \text{ by (i)}} \frac{\partial\delta_\varepsilon(t)}{\partial\varepsilon}\,dt = d\big(L_{\gamma_\varepsilon(1)^{-1}}\big)\frac{\partial}{\partial\varepsilon}\gamma_\varepsilon(1).$$

∎

Remark. Alternatively, one can use the "variational formula":

$$\frac{\partial}{\partial\varepsilon}\Phi_t^{V_\varepsilon}(p) = \int_0^t d\big(\Phi_{t-s}^{V_\varepsilon}\big)_{\Phi_s^{V_\varepsilon}(p)}\underbrace{\Big(\frac{\partial}{\partial\varepsilon}V_\varepsilon\big(\Phi_s^{V_\varepsilon}(p)\big)\Big)}\,ds \in T_{\Phi_t^{V_\varepsilon}(p)}M$$

$$\in T_{\Phi_s^{v_\varepsilon}(p)} M$$

$$\left(\begin{array}{l} v_\varepsilon(p) \in T_p M \\ \frac{\partial}{\partial \varepsilon} v_\varepsilon(p) \in T_{v_\varepsilon(p)} T_p M \cong T_p M \end{array} \right)$$

where $V_\varepsilon = V(\varepsilon, \cdot)$ is a time-dependent vec. field. on a manifold M.
(Use this with time-dep. vect. field $W^\varepsilon(t, \cdot) = (D\gamma_\varepsilon(t))^R$.)

Cor. If G is a Lie grp with Lie algebra $\mathfrak{g}$,

$$\tilde{G} \underset{\substack{\uparrow \\ \text{as in} \\ \text{introduction}}}{\cong} P(G)/P(G)_0 \underset{\substack{\uparrow \\ \text{induced by} \\ \text{homeo. } D}}{\cong} P(\mathfrak{g})/P(\mathfrak{g})_0$$

We now give an alternative proof of:

Lie II: G 1-connected, H connected Lie grp.
Any morphism $\phi : \mathfrak{g} \to \underline{h}$ can be integrated to a morphism $\Phi : G \to H$.

Pf. We can assume $\tilde{H} \cong H$ (otherwise just precompose with covering projection).

Define $f : P(\mathfrak{g}) \to P(\underline{h})$, $f(\gamma) = \phi \circ \gamma$.

Since ϕ is a morphism of Lie algebras,

$$\operatorname{ad} f(\gamma) \circ \phi = \phi \circ \operatorname{ad} \gamma \qquad \big([\phi \circ \gamma(t), \phi(X)] = \phi[\gamma(t), X] \big)$$

and hence

$$B_{f(\gamma)} \circ \phi = \phi \circ B_\gamma$$

$$B_{f(\gamma)}'(t)\, B_{f(\gamma)}(t)^{-1} \cdot \phi(X) = [\phi \circ \gamma(t), \phi(X)] = \phi[\gamma(t), X]$$

which implies (by big lemma (ii)) that
if $\gamma_0 \sim \gamma_1$ then $f(\gamma_0) \sim f(\gamma_1)$.

Hence f induces a homomorphism

the twisted product

$$f(\gamma_1 \overset{\downarrow}{\cdot} \gamma_2) = f(\gamma_1) \cdot f(\gamma_2)$$

$$\begin{tikzcd} \Phi : P(\mathfrak{g})/P(\mathfrak{g})_0 \arrow[r] \arrow[d, equal] & P(\mathfrak{h})/P(\mathfrak{h})_0 \arrow[d, equal] \\ G & H \end{tikzcd}$$

and also:

$\Phi([\gamma_1]\cdot[\gamma_2]) =$
$= \Phi[\gamma_1]\cdot\Phi[\gamma_2]$

Φ integrates ϕ i.e. $d\Phi_e = \phi$:

Consider the exponential map: $\overline{\exp} : \mathfrak{g} \to P(\mathfrak{g})/P(\mathfrak{g})_0$

$$\mathfrak{g} \xrightarrow{\exp} G \longrightarrow P(G)/P(G)_0 \xrightarrow{D} P(\mathfrak{g})/P(\mathfrak{g})_0$$

$g \mapsto$ class of any path from e to g

$:= \phi_t^{X^R}(e)$

Given $X \in \mathfrak{g}$, define $\gamma_X(t) = \exp(tX)$, and:

$$(D\gamma_X)(t) = d(R_{\exp(tX)^{-1}})\,\underbrace{\gamma_X'(t)}_{X_{\gamma_X(t)} = d(R_{\exp(tX)})_e X} = X$$

Hence $\overline{\exp}_G : \mathfrak{g} \to P(\mathfrak{g})/P(\mathfrak{g})_0$ takes $X \in \mathfrak{g}$ to the class of constant path into X.

naturality of exp translates to that of $\overline{\exp}$.

$$\rightsquigarrow \overline{\exp}_H(d\Phi_e(X)) = \Phi(\overline{\exp}_G(X))$$

$=$ class of constant path into $\phi(X)$ $= \overline{\exp}_H(\phi(X))$.

Now use the fact that exp is a local diffeo at e.